Functions of a Complex Variable, Operational Calculus, and Stability Theory

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Mir Publishers Moscow

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ФУНКЦИИ КОМПЛЕКСНОГО ПЕРЕМЕННОГО ОПЕРАЦИОННОЕ ИСЧИСЛЕНИЕ ТЕОРИЯ УСТОЙЧИВОСТИ

ИЗДАТЕЛЬСТВО «НАУКА»

Functions of a Complex Variable, Operational Calculus, and Stability Theory

PROBLEMS AND EXERCISES

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Preface

The modern engineer often has to do with problems that require a sound mathematical background and set skills in the use of diverse mathematical methods. Expanding the mathematical outlook of the engineer contributes appreciably to new advances in technology. The theory of functions of a complex variable, operational calculus, and stability theory are most important for applications.

The present text is designed for students of engineering colleges and also for practicing engineers who feel a need to refresh their knowledge of such important areas of mathematics. Each section of the text starts out with a brief review of the essentials of theory (propositions, definitions, formulas). This is followed by examples with detailed solutions and exercises. The book contains about 1000 examples and problems to be solved by the student.

It would be impossible to acknowledge all the help, direct and indirect, that we have received in the preparation of this book, but certainly the many comments of Professor V.A. Trenogin, Head of the Mathematics Department at the Moscow Institute of Steel and Alloys, and of Associate Professor M.I. Orlov of the same department must be acknowledged. We are also greatful to members of the Applied Mathematics Department at the Kiev Engineering Construction Institute (head of department Associate Professor A.E. Zhuravel') and to B. Tkachev (Krasnodar) and

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Chapter 1

Functions of a Complex Variable

§ 1 Complex Numbers and Operations with Them

A complex number z is completely specified by a pair of real numbers x and y:

$$z = x + iy$$

(the *algebraic form* of a complex number), where $i^2 = -1$ (i is known as the *imaginary unit*). The number x is called the *real part* of the complex number and y the *imaginary part*. The notations are

$$x = \operatorname{Re} z, \quad y = \operatorname{Im} z.$$

The *conjugate* $\bar{z}$ of the complex number $z = x + iy$ is $\bar{z} = x - iy$. Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are assumed to be equal if and only if $x_1 = x_2$ and $y_1 = y_2$.

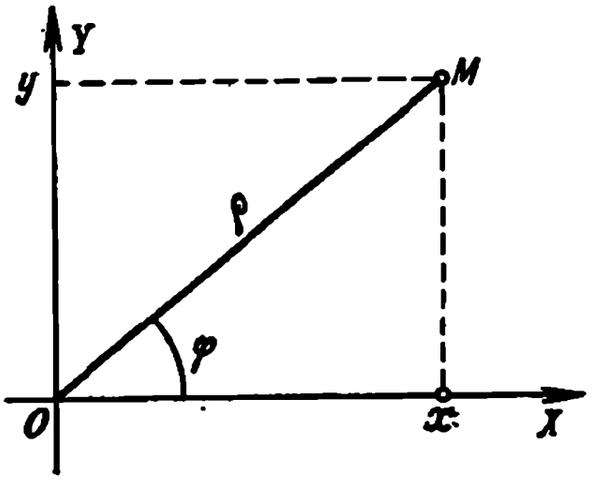


Figure 1

Any complex number $z = x + iy$ can be represented in a plane XOY (called the *complex (number) plane*, the *Gauss-Argand plane*, or the *Gaussian plane*) by either the point (x, y) (point M in Fig. 1) or the vector $\overrightarrow{OM}$ whose tail is at point O (0, 0) and tip at point M (x, y). The length ρ of vector $\overrightarrow{OM}$ is known as the *modulus* of the complex number and is denoted $|z|$, so that $\rho = |z| = (x^2 + y^2)^{1/2}$. The angle φ between the vector $\overrightarrow{OM}$ and the x axis is called the *argument* of z and is denoted $\varphi = \operatorname{Arg} z$; the argument of a complex number is determined only up to an integer mul-

tiple of 2π:

$$\operatorname{Arg} z = \arg z + 2k\pi \quad (k = 0, \pm 1, \pm 2, \ldots),$$

where $\arg z$ is the *principal value* of $\operatorname{Arg} z$, namely,

$$-\pi < \arg z < \pi,$$

with

$$\arg z = \begin{cases} \arctan(y/x) & \text{if } x > 0, \\ \pi + \arctan(y/x) & \text{if } x < 0 \text{ and } y \geqslant 0, \\ -\pi + \arctan(y/x) & \text{if } x < 0 \text{ and } y < 0, \\ \pi/2 & \text{if } x = 0 \text{ and } y > 0, \\ -\pi/2 & \text{if } x = 0 \text{ and } y < 0. \end{cases} \tag{1.1}$$

The following formulas are true:

$$\tan(\operatorname{Arg} z) = \frac{y}{x}, \quad \sin(\operatorname{Arg} z) = \frac{y}{\sqrt{x^2+y^2}},$$

$$\cos(\operatorname{Arg} z) = \frac{x}{\sqrt{x^2+y^2}}.$$

Two complex numbers z_1 and z_2 are considered equal if and only if their moduli are equal and their arguments are equal or differ by an ineger multiple of 2π:

$$|z_1| = |z_2|, \ \operatorname{Arg} z_1 = \operatorname{Arg} z_2 + 2\pi n \ (n = 0, \pm 1, \pm 2, \ldots).$$

Suppose we have two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

1. The *sum* $z_1 + z_2$ of z_1 and z_2 is a complex number defined thus:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

2. The *difference* $z_1 - z_2$ of z_1 and z_2 is defined as

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

3. The *product* z_1z_2 of z_1 and z_2 is given by the following formula:

$$z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

This definition yields, for one,

$$z\bar{z} = x^2 + y^2 = |z|^2.$$

4. The *quotient* z_1/z_2 of the division of z_1 by z_2 ($z_2 \neq 0$) is a complex number z such that $zz_2 = z_1$. For z_1/z_2 we have

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{|z_2|^2}. \tag{1.2}$$

Here we used the formula $z_2^{-1} = \bar{z}_2/|z_2|^2$. We can also write (1.2) as

$$\frac{z_1}{z_2} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + i\,\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}.$$

The real part $\operatorname{Re} z$ and the imaginary part $\operatorname{Im} z$ of a complex number z can be expressed in terms of z and $\bar{z}$ thus:

$$\operatorname{Re} z = \frac{\bar{z}+z}{2}, \quad \operatorname{Im} z = i\,\frac{\bar{z}-z}{2} = \frac{z-\bar{z}}{2i}.$$

Example 1. Show that $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.

Proof. By definition,

$$\overline{z_1+z_2} = (x_1+x_2) - i\,(y_1+y_2) = (x_1 - iy_1) + (x_2 - iy_2) = \bar{z}_1 + \bar{z}_2.$$

1. Prove that

(a) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$; (b) $\overline{z_1z_2} = \bar{z}_1\bar{z}_2$; (c) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$;

(d) $\overline{\bar{z}_1 + \bar{z}_2} = z_1 + z_2$.

Example 2. Find the real solutions of the equation

$$(4 + 2i)\,x + (5 - 3i)\,y = 13 + i.$$

Solution. Separate the real and imaginary parts of the left-hand side of this equation, i.e. write $(4x + 5y) + i\,(2x - 3y) = 13 + i$. This, according to the definition of equality of two complex numbers, yields

$$\begin{cases} 4x+5y=13, \\ 2x-3y=1. \end{cases}$$

Solving this system of simultaneous equations, we find that

$$x = 2, \quad y = 1.$$

Find the real solutions of the following equations:

2. $(3x - i)(2 + i) + (x - iy)(1 + 2i) = 5 + 6i$.

3. $(x - iy)(a - ib) = i^5$, where a and b are given real numbers, and $|a| \neq |b|$.

4. $\frac{1}{z-i} + \frac{2+i}{1+i} = \sqrt{2}$, where $z = x + iy$.

5. Write the complex number $\frac{1}{(a+ib)^2}+\frac{1}{(a-ib)^2}$ in algebraic form.

6. Prove that $\frac{1\sqrt{1+x^2}+ix}{x-i\sqrt{1+x^2}}=i$ (x is a real number).

7. Express x and y in terms of u and v if $\frac{1}{x+iy}+\frac{1}{u+iv}=1$ (x, y, u, and v are real numbers).

8. Find every complex number that satisfies the condition $\bar{z}=z^2$.

Example 3. Find the modulus and the argument of the complex number

$$z=-\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}.$$

Solution. We have

$$x=-\sin\frac{\pi}{8}<0, \quad y=-\cos\frac{\pi}{8}<0.$$

The principal value of the argument, according to (1.1), is

$$\begin{aligned}\arg z &= -\pi+\arctan\left(\cot\frac{\pi}{8}\right)\\ &= -\pi+\arctan\left[\tan\left(\frac{\pi}{2}-\frac{\pi}{8}\right)\right]\\ &= -\pi+\arctan\left(\tan\frac{3\pi}{8}\right)=-\pi+\frac{3}{8}\pi=-\frac{5}{8}\pi.\end{aligned}$$

Hence,

$$\operatorname{Arg} z=-\frac{5}{8}\pi+2k\pi \quad (k=0,\ \pm1,\ \pm2,\ \ldots),$$

$$|z|=\sqrt{\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}}=1.$$

9. In the problems below find the modulus and the principal value of the argument of the following complex numbers:

(a) $z=4+3i$; (b) $z=-2+2\sqrt{3}i$;

(c) $z=-7-i$; (d) $z=-\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$;

(e) $z=4-3i$; (f) $z=\cos\alpha-i\sin\alpha$ $\left(\pi<\alpha<\frac{3}{2}\pi\right)$.

Each complex number $z = x + iy$ $(z \neq 0)$ can be written in *polar form*:

$z = \rho(\cos\varphi + i\sin\varphi)$, where $\rho = |z|$, $\varphi = \operatorname{Arg} z$.

Example 4. Write the complex number

$$z = -1 - i\sqrt{3}$$

in polar form.

Solution. We have

$$|z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2; \quad \tan\varphi = \frac{-\sqrt{3}}{-1} = \sqrt{3},$$

$$\varphi = -\frac{2}{3}\pi.$$

Hence,

$$-1 - i\sqrt{3} = 2\left[\cos\left(-\frac{2}{3}\pi\right) + i\sin\left(-\frac{2}{3}\pi\right)\right].$$

Example 5. Find the real roots of the equation

$$\cos x + i\sin x = \frac{1}{2} + \frac{3}{4}i.$$

Solution. This equation has no roots. Indeed, the equation is equivalent to two equations: $\cos x = 1/2$ and $\sin x = 3/4$. But these equations cannot be satisfied simultaneously, since $\cos^2 x + \sin^2 x = 13/16$, which is impossible for any x.

Any complex number $z \neq 0$ can be written in exponential form:

$$z = \rho e^{i\varphi}, \quad \text{where} \quad \rho = |z|, \quad \varphi = \operatorname{Arg} z.$$

Example 6. Find all complex numbers $z \neq 0$ that satisfy the condition $z^{n-1} = \bar{z}$.

Solution. Let $z = \rho e^{i\varphi}$. Then $\bar{z} = \rho e^{-i\varphi}$. Since according to the hypothesis

$$\rho^{n-1} e^{i(n-1)\varphi} = \rho e^{-i\varphi}, \quad \text{or} \quad \rho^{n-2} e^{in\varphi} = 1,$$

we find that $\rho^{n-2} = 1$, i.e. $\rho = 1$, and $in\varphi = 2k\pi i$, i.e. $\varphi = 2k\pi/n$ $(k = 0, 1, \ldots, n-1)$. Hence,

$$z_k = e^{i(2\pi k/n)} \quad (k = 0, 1, 2, \ldots, n-1).$$

10. Represent the following complex numbers in polar form:

(a) -2; (b) $2i$; (c) $-\sqrt{2} + i\sqrt{2}$;

(d) $1 - \sin\alpha + i\cos\alpha \left(0 < \alpha < \frac{\pi}{2}\right)$;

(e) $\frac{1 + \cos\alpha + i\sin\alpha}{1 + \cos\alpha - i\sin\alpha} \left(0 < \alpha < \frac{\pi}{2}\right)$;

and in exponential form:

(f) -2; (g) i; (h) $-i$; (i) $-1-i\sqrt{3}$;

(j) $\sin\alpha - i\cos\alpha$ $(\pi/2 < \alpha < \pi)$; (k) $5+3i$.

Suppose two complex numbers, z_1 and z_2, are given in polar form: $z_1 = \rho_1(\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = \rho_2(\cos\varphi_2 + i\sin\varphi_2)$.

Their product can be determined from the following formula:

$$z_1z_2 = \rho_1\rho_2[\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)],$$

i.e. if complex numbers are multiplied, their moduli are multiplied and their arguments added:

$$|z_1z_2| = |z_1||z_2|,$$
$$\operatorname{Arg}(z_1z_2) = \operatorname{Arg}z_1 + \operatorname{Arg}z_2.$$

The quotient of the division of complex number z_1 by complex number $z_2 \neq 0$ can be determined from the following formula:

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2}[\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2)],$$

i.e.

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \operatorname{Arg}\frac{z_1}{z_2} = \operatorname{Arg}z_1 - \operatorname{Arg}z_2.$$

Raising a complex number

$$z = \rho(\cos\varphi + i\sin\varphi)$$

to a positive integer n is done according to the formula

$$z^n = \rho^n(\cos n\varphi + i\sin n\varphi),$$

i.e.

$$|z^n| = |z|^n, \quad \operatorname{Arg}z^n = n\operatorname{Arg}z + 2\pi k \quad (k = 0, \pm1, \ldots).$$

This gives *de Moivre's formula*

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi.$$

Properties of moduli of complex numbers

1. $|\bar{z}| = |z|$; 2. $z\bar{z} = |z|^2$; 3. $|z_1z_2| = |z_1||z_2|$;

4. $|z^n| = |z|^n$; 5. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$;

6. $|\operatorname{Re}z| \leqslant |z|$, $|\operatorname{Im}z| \leqslant |z|$; 7. $|z_1+z_2| \leqslant |z_1| + |z_2|$; 8. $||z_1| - |z_2|| \leqslant |z_1 - z_2|$.

Example 7. Evaluate $(-1+i\sqrt{3})^{60}$.

Solution. Represent $z = -1 + i\sqrt{3}$ in polar form:

$$-1 + i\sqrt{3} = 2\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right).$$

Applying the above formula for raising to a power yields

$$(-1 + i\sqrt{3})^{60} = 2^{60}\left[\cos\left(60\cdot\frac{2}{3}\pi\right) + i\sin\left(60\cdot\frac{2}{3}\pi\right)\right]$$
$$= 2^{60}(\cos 40\pi + i\sin 40\pi) = 2^{60}.$$

Example 8. Prove that the polynomial

$$f(x) = (\cos\alpha + x\sin\alpha)^n - \cos n\alpha - x\sin n\alpha$$

is divisible by $x^2 + 1$.

Solution. Since $x^2 + 1 = (x + i)(x - i)$, de Moivre's formula yields

$$f(i) = (\cos\alpha + i\sin\alpha)^n - \cos n\alpha - i\sin n\alpha$$
$$= \cos n\alpha + i\sin n\alpha - \cos n\alpha - i\sin n\alpha = 0.$$

Similarly, $f(-i) = 0$. This means that $f(x)$ is divisible by $x^2 + 1$.

11. Prove that the polynomial

$$f(x) = x^n\sin\alpha - \lambda^{n-1}x\sin n\alpha + \lambda^n\sin(n-1)\alpha$$

is divisible by $x^2 - 2\lambda x\cos\alpha + \lambda^2$.

12. Evaluate:

(a) $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{40}$; (b) $(2-2i)^7$;

(d) $\left(\frac{1-i}{1+i}\right)^8$.

13. Prove that

$$\left(\frac{1+i\tan\alpha}{1-i\tan\alpha}\right)^n = \frac{1+i\tan n\alpha}{1}$$

14. Prove that if

$(\cos\alpha + i\sin\alpha)^n = 1$, then

15. Using de Moivre's formula, express in terms of powers of $\sin\varphi$ and $\cos\varphi$ the following functions of integer multiples of φ:

(a) $\sin 3\varphi$; (b) $\cos 3\varphi$; (c) $\sin 4\varphi$; (d) $\cos 4\varphi$; (e) $\sin 5\varphi$; and (f) $\cos 5\varphi$.

An nth root of a complex number z has n different values given by the formula

$$\sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right),$$

where $k = 0, 1, 2, \ldots, n-1$, and $\varphi = \arg z$. The points corresponding to the values of $\sqrt[n]{z}$ are vertices of the regular n-gon inscribed in the circle of radius $R = \sqrt[n]{|z|}$ centered at the origin of coordinates.

The nth root of a real number a also has n different values; among these there are two real roots, one, or no real root, depending on whether n is even or odd and the sign of a.

Example 9. Find all values of $\sqrt[4]{1-i}$.

Solution. Reduce the complex number $1-i$ to polar form:

$$1-i = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right].$$

Hence,

$$\sqrt[4]{1-i} = \sqrt[8]{2}\left(\cos\frac{-\frac{\pi}{4}+2k\pi}{4} + i\sin\frac{-\frac{\pi}{4}+2k\pi}{4}\right).$$

Putting $k=0, 1, 2, 3$ yields

$$(k=0)\quad \sqrt[4]{1-i} = \sqrt[8]{2}\left(\cos\frac{\pi}{16} - i\sin\frac{\pi}{16}\right),$$

$$(k=1)\quad \sqrt[4]{1-i} = \sqrt[8]{2}\left(\cos\frac{7}{16}\pi + i\sin\frac{7}{16}\pi\right),$$

$$(k=2)\quad \sqrt[4]{1-i} = \sqrt[8]{2}\left(\cos\frac{15}{16}\pi + i\sin\frac{15}{16}\pi\right),$$

$$(k=3)\quad \sqrt[4]{1-i} = \sqrt[8]{2}\left(\cos\frac{23}{16}\pi + i\sin\frac{23}{16}\pi\right).$$

In the following exercises find all values of the root:

16. (a) $\sqrt[4]{-1}$; (b) $\sqrt{i}$; (c) $\sqrt[3]{i}$; (d) $\sqrt[4]{-i}$.

17. (a) $\sqrt{1}$; (b) $\sqrt[3]{-1+i}$; (c) $\sqrt{2-2\sqrt{3}i}$.

18. $\sqrt[5]{\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}$.

Example 10. What set of points in the complex z plane is defined by the condition

$$\operatorname{Im} z^2 > 2?$$

Solution. Let $z = x + iy$. Then

$$z^2 = (x + iy)^2 = (x^2 - y^2) + i2xy.$$

Hence $\operatorname{Im} z^2 = 2xy$. By hypothesis, $2xy > 2$, or $xy > 1$. This inequality defines the set of points in the first and third quadrants, respectively, above and below the hyperbola $xy = 1$.

Example 11. What set of points in the complex z plane is defined by the condition

$$-\frac{\pi}{2} \leqslant \arg(z + 1 - i) \leqslant \frac{3}{4}\pi?$$

Solution. The complex number

$$z + 1 - i = z - (-1 + i)$$

is represented by a vector whose tail is at point $-1 + i$ and tip at point z. The angle between this vector and the x axis is $\arg(z + 1 - i)$, and it varies from $-\pi/2$ to $3\pi/4$. Hence, the above inequality defines an angle between the straight lines that emerge at point $-1 + i$ and form angles of $-\pi/2$ and $3\pi/4$ radians with the x axis (Fig. 2).

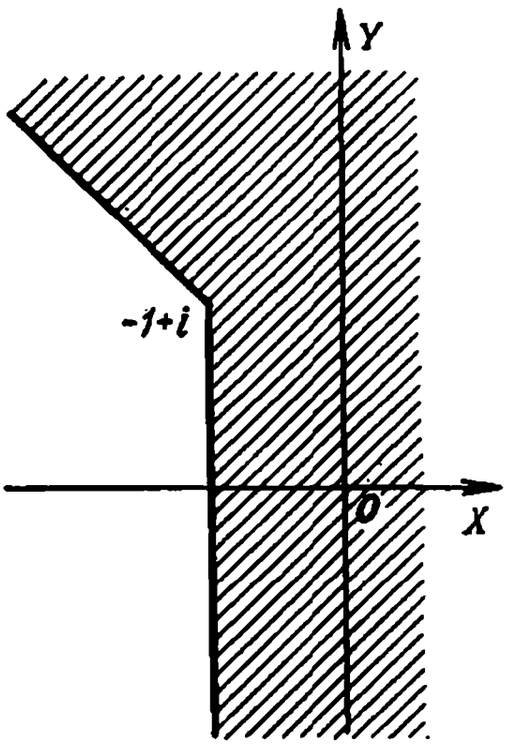


Figure 2

Example 12. What region is given by the condition $|z| + \operatorname{Re} z < 1$?

Solution. Let $z = \rho(\cos\varphi + i\sin\varphi)$. Then $|z| = \rho$ and $\operatorname{Re} z = \rho\cos\varphi$. By hypothesis, $\rho + \rho\cos\varphi < 1$. Whence, $\rho < 1/(1 + \cos\varphi)$. This condition specifies a set of points lying inside the region bounded by the curve $\rho = 1/(1 + \cos\varphi)$ (the equation of the parabola in polar coordinates).

In the problems below find the sets of points in the complex z plane that are defined by the given conditions:

19. (a) $|z| \geqslant 2$; (b) $\frac{1}{|z|} \geqslant 1$, $z \neq 0$; (c) $\left|\frac{1}{z}\right| \leqslant 2$, $z \neq 0$.

20. (a) $|z - 5i| = 8$; (b) $|z - 1 - i| \leqslant 4$.

21. (a) $1 < |z+i| < 2$, $\frac{\pi}{4} < \arg z < \frac{\pi}{2}$; (b) $2 < |z| < 3$, $\frac{\pi}{8} < \arg z < \frac{4}{3}\pi$.

22. (a) $\left|\frac{z-1}{z+1}\right| \leqslant 1$; (b) $0 \leqslant \operatorname{Im} z \leqslant 1$.

23. (a) $1 \leqslant |z+2+i| \leqslant 2$; (b) $|z-1| < |z-i|$; (c) $1 < \operatorname{Re} z < 2$.

24. $|z-a| < |1-a\bar{z}|$ (a is real and $|a| < 1$).

25. (a) $|z| > 2 + \operatorname{Im} z$; (b) $|z| - \operatorname{Re} z \leqslant 0$.

26. $\operatorname{Im} \bar{z}^2 < 1$.

27. $4 \leqslant |z-1| + |z+1| \leqslant 8$.

28. (a) $\operatorname{Im}\left(\frac{1}{z}\right) < -\frac{1}{2}$; (b) $\frac{1}{4} < \operatorname{Re}\left(\frac{1}{\bar{z}}\right) + \operatorname{Im}\left(\frac{1}{\bar{z}}\right) < \frac{1}{2}$.

Example 13. Find the curve whose equation is $|z+c| + |z-c| = 2a$, where c and a are positive real numbers such that $a > c$.

Solution. $|z+c|$ is the distance between points z and $-c$, and $|z-c|$ is the distance between points z and c. By hypothesis, the sum of the distances from point z to the two given points, $z_1 = -c$ and $z_2 = c$, is constant. This means that point z lies on an ellipse. The equation of this ellipse is $x^2/a^2 + y^2/b^2 = 1$, where $b^2 = a^2 - c^2$.

Example 14. Find the curve whose equation is $\operatorname{Re}(1/z) = 1/4$.

Solution. Let $z = x + iy$. We have

$$\operatorname{Re}\left(\frac{1}{z}\right) = \frac{\frac{1}{\bar{z}} + \frac{1}{z}}{2} = \frac{z+\bar{z}}{2z\bar{z}} = \frac{x}{x^2+y^2}.$$

By hypothesis,

$$\frac{x}{x^2+y^2} = \frac{1}{4}, \quad \text{or} \quad x^2+y^2-4x = 0.$$

This is a circle: $(x-2)^2 + y^2 = 4$.

29. What curve constitutes the set of all points $z = -2 + iy$ if y assumes any real value?

30. What curve constitutes the set of all points $z = x + 2i$ if x assumes any real value?

Specify the curves represented by the following equations:

31. (a) $\operatorname{Im} z^2 = 2$; (b) $\operatorname{Re} \bar{z}^2 = 1$; (c) $\operatorname{Im} (1/z) = 1/2$.

32. (a) $\operatorname{Re} (1/\bar{z}) = 1$; (b) $\operatorname{Im} (\overline{z^2 - \bar{z}}) = 2 - \operatorname{Im} z$.

33. $z^2 + \bar{z}^2 = 1$.

34. $2z\bar{z} + (2 + i) z + (2 - i) \bar{z} = 2$.

35. (a) $|z - i| + |z + i| = 4$; (b) $|z - i| - |z + i| = 2$.

36. (a) $|z| - 3 \operatorname{Im} z = 6$; (b) $3|z| - \operatorname{Re} z = 12$.

37. (a) $|z - 2| = |1 - 2\bar{z}|$; (b) $|z - z_1| = |z - z_2|$; (c) $\operatorname{Re} (z^2 - \bar{z}) = 0$; (d) $\operatorname{Re} (1 + z) = |z|$.

Example 15. Write the equation of a straight line,

$$Ax + By + C = 0, \tag{1.3}$$

in complex form.

Solution. Let $z = x + iy$ and $\bar{z} = x - iy$. Then $x = (\bar{z} + z)/2$ and $y = i(\bar{z} - z)/2$. Substitution of these formulas into Eq. (1.3) yields

$$A(\bar{z} + z) + Bi(\bar{z} - z) + 2C = 0,$$

or

$$(A + iB)\bar{z} + (A - iB) z + 2C = 0. \tag{1.4}$$

Equation (1.4) can be written as

$$a\bar{z} + \bar{a}z + 2C = 0,$$

where $a = A + iB$.

Example 16. Write the equation of the circle

$$x^2 + y^2 + 2x + 2y = 0 \tag{1.5}$$

in complex form.

Solution. We have

$$x^2 + y^2 = |z|^2 = z\bar{z}, \quad 2x = \bar{z} + z, \quad 2y = i(\bar{z} - z).$$

Substitution into Eq. (1.5) yields

$$z\bar{z} + \bar{z} + z + i(\bar{z} - z) = 0,$$

or

$$z\bar{z} + (1 - i) z + (1 + i) \bar{z} = 0.$$

Example 17. What line in the complex number plane is given by the equation

$$z\bar{z} + i(z - \bar{z}) - 2 = 0? \tag{1.6}$$

Solution. Let $z = x + iy$. We have $\bar{z} = x - iy$ and $z\bar{z} = x^2 + y^2$. Equation (1.6) becomes

$$x^2 + y^2 - 2y - 2 = 0, \quad \text{or} \quad x^2 + (y - 1)^2 = 3.$$

This is a circle of radius $R = \sqrt{3}$ centered at point (0, 1).

Write the following equations in complex form:

38. (a) The equations of the x and y axes; (b) of the straight line $y = x$; (c) of the straight line $y = kx + b$, where k and b are real numbers.

39. (a) The equation of the equilateral hyperbola $x^2 - y^2 = a^2$; (b) of the circle $x^2 + y^2 + 2x = 0$.

Various Problems

Solve the following equations:

40. $z^3 + 3z^2 + 3z + 3 = 0$.

41. $z^4 - 4z^3 + 6z^2 - 4z - 15 = 0$.

42. Find the complex number z whose image in the complex number plane is a point on the segment z_1z_2 that is twice as far from z_2 than from z_1.

43. Into what vector will the vector $a + ib$ be mapped under a mirror reflection with respect to the bisectrix of the first quadrant?

44. Into what vector will the vector $-\sqrt{3} + 3i$ be mapped under a rotation through 90°?

45. Into what vector will the vector $-\sqrt{3} - i$ be mapped under a rotation through 120°?

46. Find the angle about which the vector $4 - 3i$ must be rotated to obtain the vector $-5/\sqrt{2} + i5/\sqrt{2}$.

47. Find the angle about which the vector $3\sqrt{2} + i2\sqrt{2}$ must be rotated to obtain the vector $-5 + i$.

Solve the following equations:

48. $(x + i)^n - (x - i)^n = 0$ (x is real).

49. $\cos x + i \sin x = \sin x + i \cos x$.

50. Find the vector into which the vector $z = 3 + 4i$ is mapped under a rotation through 45° and a stretching by a factor of 2.

51. The center of a square is at point $z_0 = 1 + i$ and one of the square's vertices at point $z_1 = 1 - i$. At what points are the other three vertices?

52. Let $z_1, z_2, \ldots, z_n$ be the roots of the equation $z^n - 1 = 0$ ($n > 1$). Prove that $z_1 + z_2 + \ldots + z_n = 0$.

Find the following sums:

53. (a) $\sin x + \sin 2x + \ldots + \sin nx$;

(b) $\cos x + \cos 2x + \ldots + \cos nx$.

54. (a) $\sin x + \sin 3x + \ldots + \sin (2n - 1) x$;
(b) $\cos x + \cos 3x + \ldots + \cos (2n - 1) x$.

§ 2 Functions of a Complex Variable

We say that a *function* $w = f(z)$ is defined in a domain D if to each point $z \in D$ there corresponds one (a single-valued function) or several (a multiple-valued function) values of w. We can also say that the function $w = f(z)$ maps the points in the complex z plane into points in the complex w plane.

Let $z = x + iy$ and $w = u + iv$. The dependence $w = f(z)$ of the dependent complex variable w on the independent complex variable z can then be described by specifying two real-valued functions u and v of the real variables x and y:

$$u = u(x, y), \quad v = v(x, y).$$

Example 1. Let $w = z^3 - i\bar{z}$. Putting $z = x + iy$ and $w = u + iv$, we find that

$$\begin{aligned} u + iv &= (x + iy)^3 - i(x - iy) \\ &= (x^3 - 3xy^2 - y) + i(3x^2y - y^3 - x). \end{aligned}$$

Hence, the equation $w = z^3 - i\bar{z}$ is equivalent to the following two equations:

$$\begin{cases} u = x^3 - 3xy^2 - y, \\ v = 3x^2y - x - y^3. \end{cases}$$

Find the real and imaginary parts of the following functions:

55. (a) $w = \bar{z} - iz^2$; (b) $w = z^2 + i$; (c) $w = i - z^3$; (d) $w = 1/\bar{z}$; (e) $w = (iz + 1)/(1 + \bar{z})$; (f) $w = \bar{z}/z$.

In the problems that follow find the points into which the given points are mapped under the specified transformations:

56. (a) $z_0 = -i$; $w = z^2$; (b) $z_0 = 1 - i$, $w = (z - i)^2$;

(c) $z_0 = 1$, $w = \dfrac{1}{z - i}$; (d) $z_0 = 2 + 3i$, $w = \bar{z}/z$.

Suppose that a curve in the complex z plane is given by the equation $F(x, y) = 0$. To find the equation specifying the map $\Phi(u, v) = 0$ of this curve in the complex w plane under the transformation $w = f(z) = u + iv$ we must

eliminate x and y from the equations

$$\begin{cases} u = u(x, y), \\ v = v(x, y), \\ F(x, y) = 0. \end{cases}$$

If a curve is specified parametrically, i.e.

$$\left.\begin{aligned} x &= x(t), \\ y &= y(t) \end{aligned}\right\}, \quad \text{or} \quad z = z(t) = x(t) + iy(t),$$

then the parametric equation of its map under a transformation $w = f(z) = u + iv$ is

$$\left.\begin{aligned} u &= u[x(t), y(t)] = U(t), \\ v &= v[x(t), y(t)] = V(t). \end{aligned}\right\}$$

Example 2. What is the curve into which the unit circle $|z| = 1$ is mapped under the transformation $w = z^2$?

Solution. Since by hypothesis $|z| = 1$, we find that

$$|w| = |z|^2 = 1.$$

Hence, the circle $|z| = 1$ in the complex z plane is mapped into the circle $|w| = 1$ in the complex w plane, with the latter being traversed twice. This follows from the fact that since $w = z^2$, we conclude that $\operatorname{Arg} w = 2 \operatorname{Arg} z + 2k\pi$, so that when the point z traverses a complete circle $|z| = 1$ its image traverses the circle $|w| = 1$ twice.

Example 3. Find the map of the circle

$$z = R \cos t + iR \sin t \quad (0 \leqslant t < 2\pi)$$

under the transformation $w = z/\bar{z}$.

Solution. Let $z = x + iy$. We can write the given equation of the circle as

$$x = R \cos t, \quad y = R \sin t \quad (0 \leqslant t < 2\pi).$$

We separate the real and imaginary parts of the function $w = u + iv$. This yields

$$u + iv = \frac{z}{\bar{z}} = \frac{z^2}{z\bar{z}} = \frac{x^2 - y^2}{x^2 + y^2} + i\,\frac{2xy}{x^2 + y^2}.$$

Whence

$$u = \frac{x^2 - y^2}{x^2 + y^2}, \quad v = \frac{2xy}{x^2 + y^2}.$$

Substituting $x = R\cos t$ and $y = R\sin t$ into u and v yields the parametric equations of the map of the circle:

$$\begin{cases} u = \dfrac{\cos^2 t - \sin^2 t}{\cos^2 t + \sin^2 t} = \cos 2t, \\ v = \dfrac{2\cos t \sin t}{\cos^2 t + \sin^2 t} = \sin 2t \end{cases} \qquad (*)$$

or $u^2 + v^2 = 1$. Hence, the map is a unit circle traversed twice, which follows from the fact that $0 \leqslant t < 2\pi$ and from (*).

57. Find the maps in the complex w plane of the following curves in the complex z plane under the transformation $w = 1/z$.

(a) $|z| = 1/2$; (b) $\operatorname{Re} z = 0$; (c) $\arg z = 3\pi/4$; (d) $\arg z^2 = -\pi/2$; (e) $\operatorname{Re} z = \operatorname{Im} z$; (f) $|z| = z$.

58. Find the maps of the x and y axes under the following transformations:

(a) $w = (z + 1)/(z - 1)$; (b) $w = 1 + 1/z$.

The basic elementary functions of a complex variable

1. The *fractional rational function*

$$w = \frac{a_0 z^n + a_1 z^{n-1} + \ldots + a_n}{b_0 z^m + b_1 z^{m-1} + \ldots + b_m};$$

for one, the polynomial

$$w = a_0 z^n + a_1 z^{n-1} + \ldots + a_n$$

is a rational function.

2. The *exponential function* e^z is defined as the sum of the power series

$$e^z = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots$$

absolutely convergent in the entire complex plane. It possesses the following properties:

(a) $e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$, where z_1 and z_2 are any complex numbers;

(b) $e^{z + 2k\pi i} = e^z$ ($k = 0, \pm 1, \pm 2, \ldots$), i.e. e^z is a periodic function with a period of $2\pi i$.

3. The *trigonometric functions* $\sin z$ and $\cos z$ are defined by the power series

$$\sin z = z - \frac{z^3}{3!} + \ldots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \ldots,$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots + (-1)^n \frac{z^{2n}}{(2n)!} + \ldots,$$

which are absolutely convergent for all complex values of z. The functions $\sin z$ and $\cos z$ are periodic with a real period 2π and possess only real zeros at $z = k\pi$ and $z = \pi/2 + k\pi$, respectively, where $k = 0, \pm 1, \pm 2, \ldots$.

The functions e^z, $\sin z$, $\cos z$ are related via *Euler's formulas*:

$$e^{iz} = \cos z + i \sin z, \quad e^{-iz} = \cos z - i \sin z, \tag{2.1}$$

whence

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}. \tag{2.2}$$

The functions $\tan z$ and $\cot z$ are defined thus:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}. \tag{2.3}$$

All the formulas of trigonometry remain valid for trigonometric functions of a complex variable.

4. The *hyperbolic functions* $\sinh z$, $\cosh z$, $\tanh z$, and $\coth z$ are defined thus:

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}, \tag{2.4}$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}. \tag{2.5}$$

5. The trigonometric and hyperbolic functions are coupled through the following relations:

$$\sin z = -i \sinh iz, \qquad \sinh z = -i \sin iz,$$
$$\cos z = \cosh iz, \qquad \cosh z = \cos iz,$$
$$\tan z = -i \tanh iz, \qquad \tanh z = -i \tan iz,$$
$$\cot z = i \coth iz, \qquad \coth z = i \cot iz.$$

6. The *logarithmic function* $\operatorname{Ln} z$, with $z \neq 0$, is defined as the inverse of the exponential function, and

$$\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z = \ln |z| + i \arg z + 2k\pi i \tag{2.6}$$

$$(k = 0, \pm 1, \pm 2, \ldots).$$

This is a multiple-valued function. The principal value of Ln z is the value with $k = 0$; it is denoted ln z:

$$\ln z = \ln |z| + i \arg z.$$

Obviously,

$$\operatorname{Ln} z = \ln z + 2k\pi i, \quad k = 0, \pm 1, \pm 2, \ldots.$$

The following relations are true:

$$\operatorname{Ln}(z_1 \cdot z_2) = \operatorname{Ln} z_1 + \operatorname{Ln} z_2,$$
$$\operatorname{Ln}(z_1/z_2) = \operatorname{Ln} z_1 - \operatorname{Ln} z_2.$$

7. The *inverse trigonometric functions* Arc sin z, Arc cos z, Arc tan z, and Arc cot z (another notation is $\sin^{-1} z$, $\cos^{-1} z$, $\tan^{-1} z$, and $\cot^{-1} z$) are respectively defined as the inverses of sin w, cos w, tan w, and cot w. For instance, if $z = \sin w$, we say that w is the arcsine of z and write $w = \operatorname{Arc}\sin z$.

All these functions are multiple-valued and can be expressed in terms of logarithmic functions thus:

$$\operatorname{Arc}\sin z = -i \operatorname{Ln}\left(iz + \sqrt{1 - z^2}\right); \tag{2.7}$$

$$\operatorname{Arc}\cos z = -i \operatorname{Ln}\left(z + \sqrt{z^2 - 1}\right); \tag{2.8}$$

$$\operatorname{Arc}\tan z = -\frac{i}{2} \operatorname{Ln}\frac{1 + iz}{1 - iz}; \tag{2.9}$$

$$\operatorname{Arc}\cot z = -\frac{i}{2} \operatorname{Ln}\frac{z + i}{z - i}. \tag{2.10}$$

The principal values of the inverse trigonometric functions can be obtained through the corresponding principal values of logarithmic functions.

8. The *general power function* $w = z^a$, where $a = \alpha + i\beta$ is an arbitrary complex number, is defined thus:

$$z^a = e^{a \operatorname{Ln} z}.$$

Generally, this function is multiple-valued; its principal value is

$$z^a = e^{a \operatorname{Ln} z}.$$

9. The *general exponential function* $w = a^z$ ($a \neq 0$ is an arbitrary complex number) is defined thus:

$$a^z = e^{z \operatorname{Ln} a}.$$

The principal value of this multiple-valued function is $a^z = e^{z \operatorname{Ln} a}$.

Separate the real and imaginary parts of the following functions:

59. (a) $w = 2z - 1$; (b) $w = z + z^2$; (c) $w = z^{-1}$.

60. (a) $w = e^{-z}$; (b) $w = e^{\bar{z}^2}$; (c) $w = \sin z$; (d) $w = \cosh(z - i)$.

61. (a) $w = 2^{z^2}$; (b) $w = \sinh z$; (c) $w = \tan z$.

Example 4. Find the modulus of the function $w = \sin z$ at

$$z = \pi + i \ln (2 + \sqrt{5}).$$

Solution. Let $z = x + iy$. Then

$$w = \sin x \cosh y + i \sinh y \cos x.$$

The modulus of $\sin z$ is

$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x}$$

$$= \sqrt{\sin^2 x \cosh^2 y + \sinh^2 y (1 - \sin^2 x)}$$

$$= \sqrt{\sin^2 x + \sinh^2 y}.$$

Putting $z = \pi + i \ln (2 + \sqrt{5})$, we find that

$$|\sin |\pi + i \ln (2 + \sqrt{5})|| = \sinh |\ln (2 + \sqrt{5})|$$

$$= \frac{e^{\ln (2 + \sqrt{5})} - e^{-\ln (2 + \sqrt{5})}}{2} = \frac{2 + \sqrt{5} - \dfrac{1}{2 + \sqrt{5}}}{2} = 2.$$

This example shows that the trigonometric function $\sin z$ in the complex domain can assume values whose moduli are greater than unity.

In the following problems find the moduli and principal values of the arguments of the specified functions at the given points:

62. $w = \cos z$, (a) $z_1 = \pi/2 + i \ln 2$; (b) $z_2 = \pi + i \ln 2$.

63. $w = \sinh z$, $z_0 = 1 + i\pi/2$.

64. $w = ze^z$, $z_0 = i\pi$.

65. $w = \cosh^2 z$, $z_0 = i \ln 3$.

66. Find the logarithms of the following numbers:
(a) e; (b) $-i$; (c) i; (d) $-1 - i$; (e) $3 - 2i$; (f) i^i.

67. Find (a) i^i; (b) $i^{1/i}$; (c) 1^i; (d) $(-1)^{\sqrt{2}}$;

(e) $\left(\dfrac{1+i}{\sqrt{2}}\right)^{2i}$; (f) $\left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)^{1+i}$; (g) $(1 - i)^{3-3i}$.

68. Find the modulus ρ and argument φ of the following complex numbers:

(a) $\tanh \pi i$; (b) 10^i; (c) 3^{2-i}.

Example 5. Write $\operatorname{Arc\,sin}(i\pi/3)$ in algebraic form.

Solution. If in Eq. (2.7) we put $z = i\pi/3$, we find that

$$\operatorname{Arc\,sin}\frac{\pi}{3}i = -i\operatorname{Ln}\left(-\frac{\pi}{3} \pm \sqrt{1+\frac{\pi^2}{9}}\right).$$

Whence,

$$\operatorname{Arc\,sin}\frac{\pi}{3}i = -i\operatorname{Ln}\left[-\left(\frac{\pi}{3}+\sqrt{1+\frac{\pi^2}{9}}\right)\right]$$

$$= -i\left[\ln\left(\frac{\pi}{3}+\sqrt{1+\frac{\pi^2}{9}}\right)+\pi i+2k\pi i\right]$$

$$= (2k+1)\pi - i\ln\left(\frac{\pi}{3}+\sqrt{1+\frac{\pi^2}{9}}\right) \quad (k=0, \pm 1, \pm 2, \ldots$$

and

$$\operatorname{Arc\,sin}\frac{\pi}{3}i = -i\ln\left(\sqrt{1+\frac{\pi^2}{9}}-\frac{\pi}{3}\right)$$

$$= -i\left[\ln\left(\sqrt{1+\frac{\pi^2}{9}}-\frac{\pi}{3}\right)+2k\pi i\right]$$

$$= 2k\pi - i\ln\left(\sqrt{1+\frac{\pi^2}{9}}-\frac{\pi}{3}\right) \quad (k=0, \pm 1, \pm 2, \ldots).$$

Example 6. Write $\operatorname{Arc\,tan}(1+i)$ in algebraic form.

Solution. If in Eq. (2.9) we put $z = 1 + i$, we find that

$$\operatorname{Arc\,tan}(1+i) = -\frac{i}{2}\operatorname{Ln}\frac{1+i(1+i)}{1-i(1+i)} = -\frac{i}{2}\operatorname{Ln}\frac{i}{2-i}$$

$$= -\frac{i}{2}\operatorname{Ln}\left(-\frac{1}{5}+\frac{2}{5}i\right).$$

Moreover,

$$\operatorname{Ln}\left(-\frac{1}{5}+\frac{2}{5}i\right) = -\operatorname{Ln}\sqrt{5}+(2k+1)\pi i - i\operatorname{arc\,tan}2.$$

Finally,

$$\operatorname{Arc\,tan}(1+i) = -\frac{1}{2}\operatorname{arc\,tan}2+(2k+1)\frac{\pi}{2}+\frac{i}{2}\ln\sqrt{5}$$

$$(k=0, \pm 1, \pm 2, \ldots).$$

Write the following complex numbers in algebraic form:

69. (a) $e^{i\pi/4}$; (b) $\operatorname{Ln}(1-i)$.
70. (a) $\sin \pi i$; (b) $\cos \pi i$; (c) $\tan (i\pi/2)$.
71. (a) $\cot \pi i$; (b) $\operatorname{Arc}\sin i$; (c) $\operatorname{Arc}\tan (i/3)$.
72. (a) $\operatorname{Arc}\cos i$; (b) $\sinh (i\pi/2)$; (c) $\tanh \pi i$.

Example 7. Solve the equation $\sin z = 3$.
Solution. The problem can be reduced to finding

$$z = \operatorname{Arc}\sin 3.$$

If we use Eq. (2.7), we find that

$$\operatorname{Arc}\sin t = -i \operatorname{Ln}(it + \sqrt{1-t^2}).$$

We have

$$z = \operatorname{Arc}\sin 3 = -i \operatorname{Ln}(3i + \sqrt{-8});$$

or, bearing in mind that $\sqrt{-8} = \pm\sqrt{8}i$,

$$z = -i\,[\operatorname{Ln}(3+\sqrt{8})\,i],$$

$$z = -i\,[\operatorname{Ln}(3-\sqrt{8})\,i].$$

Since

$$\arg[(3\pm\sqrt{8})\,i] = \arg[(3-\sqrt{8})\,i] = \frac{\pi}{2},$$

$$|(3+\sqrt{8})\,i| = 3+\sqrt{8}, \quad |(3-\sqrt{8})\,i| = 3-\sqrt{8},$$

we see that

$$\operatorname{Ln}[(3\pm\sqrt{8})\,i] = \ln(3\pm\sqrt{8}) + \frac{\pi}{2}i + 2k\pi i,$$

where $k = 0, \pm 1, \pm 2, \ldots$. Consequently,

$$z = \frac{\pi}{2} + 2k\pi - i\ln(3\pm\sqrt{8}) \qquad (k = 0; \pm 1, \pm 2, \ldots).$$

Solve the following equations:

73. $e^{-z} + 1 = 0$.
74. $e^{z} + i = 0$.
75. $4\cos z + 5 = 0$.
76. $\sinh iz = -i$.
77. $\sin z = i\pi$.
78. $e^{ix} = \cos \pi x$, with x real.
79. $e^{2z} + 2e^{z} - 3 = 0$.
80. $\cosh z = i$.
81. (a) $\ln(z+i) = 0$; (b) $\ln(i-z) = 1$.

§ 3 The Limit of a Sequence of Complex Numbers. The Limit and Continuity of a Function of a Complex Variable

1. Suppose we have a sequence $\{z_n\}$ of complex numbers:

$$z_1, z_2, \ldots, z_n, \ldots.$$

Definition 1. A complex number a is said to be the *limit of sequence* $\{z_n\}$ if for every positive number ε we can select a term in the sequence with number $N = N(\varepsilon)$ starting from which all terms z_n in the sequence satisfy the inequality

$$|z_n - a| < \varepsilon \quad \text{for} \quad n \geqslant N(\varepsilon).$$

A sequence $\{z_n\}$ with a limit a is said to converge to a; the notation for this is $\lim\limits_{n\to\infty} z_n = a$.

To each sequence $\{z_n\}$ of complex numbers there correspond two sequences, $\{x_n\}$ and $\{y_n\}$, of real numbers; here $z_n = x_n + iy_n$, $n = 1, 2, \ldots$.

Theorem 1 *The sequence* $\{z_n = x_n + iy_n\}$ *converges to* $a = \alpha + i\beta$ *if and only if*

$$\lim_{n\to\infty} x_n = \alpha, \quad \lim_{n\to\infty} y_n = \beta.$$

Definition 2. A sequence $\{z_n\}$ is said to be *bounded* if there is a positive M such that $|z_n| \leqslant M$ for all terms z_n in this sequence.

Theorem 2 *Every convergent sequence is bounded.*

The properties of convergent sequences of complex numbers

If $\lim\limits_{n\to\infty} z_n = a$ and $\lim\limits_{n\to\infty} \tau_n = b$, then

1. $\lim\limits_{n\to\infty} (z_n \pm \tau_n) = a \pm b$;
2. $\lim\limits_{n\to\infty} (z_n \tau_n) = ab$;
3. $\lim\limits_{n\to\infty} \frac{z_n}{\tau_n} = \frac{a}{b}$ $(\tau_n \neq 0,\ b \neq 0)$.

Example 1. Prove that the sequence of the numbers

$$z_n = \frac{n-i}{n+1}, \quad n = 1, 2, \ldots,$$

has $a = 1$ as its limit.

Solution. We take an arbitrary positive ε and we show that there is a term in the sequence with number N such that $|z_n - 1| < \varepsilon$ for all $n \geqslant N$. Since

$$|z_n - 1| = \left|\frac{n-i}{n+1} - 1\right| = \left|\frac{1+i}{n+1}\right| = \frac{\sqrt{2}}{n+1},$$

the inequality $|z_n - 1| < \varepsilon$ is satisfied if $\sqrt{2}/(n+1) < \varepsilon$, i.e. when n is greater than $\sqrt{2}/\varepsilon - 1$. This means that we can take

$$N = N(\varepsilon) = \left[\frac{\sqrt{2}}{\varepsilon} - 1\right] + 1.$$

Here $[x]$ stands for the integral part of the real number x.

Example 2. Let a sequence $\{z_n\}$ have a limit a. Show that $\{|z_n|\}$ has $|a|$ as its limit. Indeed, since $\lim\limits_{n\to\infty} z_n = a$, we find that

$$\lim_{n\to\infty} |z_n - a| = 0. \tag{3.1}$$

On the other hand, for any two complex numbers z_n and a we have the following inequality (see p. 14):

$$||z_n| - |a|| \leqslant |z_n - a|. \tag{3.2}$$

Combining (3.1) and (3.2), we find that $\lim\limits_{n\to\infty} |z_n| = |a|$.

Sufficient conditions for the convergence of a sequence of complex numbers

Let $z_n = \rho_n e^{i\varphi_n}$, where $\rho_n = |z_n|$ and $\varphi_n = \arg z_n$. If $\lim\limits_{n\to\infty} \rho_n = \rho_0$ and $\lim\limits_{n\to\infty} \varphi_n = \varphi_0$, then $\lim\limits_{n\to\infty} z_n = \rho_0 e^{i\varphi_0}$.

Example 3. Prove that

$$\lim_{n\to\infty} (1 + z/n)^n = e^z, \text{ where } z = x + iy.$$

Proof. We introduce the notation

$$z_n = (1 + z/n)^n.$$

Then

$$\lim_{n\to\infty} |z_n| = \lim_{n\to\infty} \left|\left(1 + \frac{z}{n}\right)^n\right|$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{x}{n}\right)^2 + \frac{y^2}{n^2} \right]^{\frac{n}{2}}$$

$$= \lim_{n\to\infty} \left(1+\frac{x^2+y^2+2xn}{n^2}\right)^{\frac{n}{2}} = e^x.$$

Since

$$\varphi_n = \arg(1+z/n) = \operatorname{arc\,tan}\frac{y/n}{1+x/n} = \operatorname{arc\,tan}\frac{y}{n+x},$$

we see that

$$\arg z_n = \arg(1+z/n)^n = n\arg(1+z/n) = n\operatorname{arc\,tan}\frac{y}{n+x}.$$

This means that

$$\lim_{n\to\infty}\varphi_n = \lim_{n\to\infty} n\operatorname{arc\,tan}\frac{y}{n+x} = y.$$

Using the sufficient condition for convergence of a complex number sequence, we find that

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = e^x e^{iy} = e^{x+iy} = e^z,$$

which is what we set out to prove.

Example 4. Prove that the sequence of complex numbers

$$z_n = \arg\frac{(-1)^n}{n}, \quad n = 1, 2, \ldots,$$

is divergent.

Proof. Since

$$z_n = \arg\frac{(-1)^n}{n} = \begin{cases} 0 & \text{for } n \text{ even}, \\ \pi & \text{for } n \text{ odd}, \end{cases}$$

the sequence $\{z_n\}$ has the form $\pi, 0, \pi, 0, \ldots$ and, hence, has no limit.

Example 5. Let

$$x_n = 1 + \rho\cos\alpha + \rho^2\cos 2\alpha + \ldots + \rho^n\cos n\alpha,$$

where $0 < \rho < 1$, $n = 1, 2, \ldots$. Find $\lim\limits_{n\to\infty} x_n$.

Solution. We put

$$y_n = \rho\sin\alpha + \rho^2\sin 2\alpha + \ldots + \rho^n\sin n\alpha$$

and consider the limit of the sequence of the complex numbers

$$z_n = x_n + iy_n = 1 + \rho(\cos\alpha + i\sin\alpha) + \rho^2(\cos 2\alpha + i\sin 2\alpha) + \ldots + \rho^n(\cos n\alpha + i\sin n\alpha)$$

If we introduce the notation

$$t = \rho(\cos\alpha + i\sin\alpha), \quad |t| = \rho < 1,$$

de Moivre's formula and the formula for the sum of a geometric progression yield

$$z_n = 1 + t + t^2 + \ldots + t^n = \frac{1 - t^{n+1}}{1 - t}.$$

But since $|t| < 1$, we have

$$\lim_{n\to\infty} z_n = \frac{1}{1-t}.$$

Consequently,

$$\lim_{n\to\infty} x_n = \operatorname{Re}\frac{1}{1-t} = \operatorname{Re}\frac{1}{1-\rho(\cos\alpha + i\sin\alpha)}$$

$$= \frac{1-\rho\cos\alpha}{(1-\rho\cos\alpha)^2 + \rho^2\sin^2\alpha} = \frac{1-\rho\cos\alpha}{1-2\rho\cos\alpha + \rho^2}.$$

Suppose we have a sequence $\{z_n\}$ of complex numbers:

$$z_1, z_2, \ldots, z_n, \ldots.$$

If to each arbitrarily large positive M there corresponds a positive integer N such that all terms z_n of the sequence whose numbers n are greater than N satisfy the inequality $|z_n| > M$, we say that $\{z_n\}$ *converges to the point at infinity*, or simply to infinity, and write $\lim_{n\to\infty} z_n = \infty$.

If to the complex number plane we mentally add the point at infinity defined in the above manner, $z = \infty$, we obtain the *extended complex plane*, or simply the *extended* plane.

The set of all points z (including the point at infinity) that satisfy the inequality $|z| > R$, i.e. the set of all points z lying in the exterior of a circle with a fairly large radius and centered at the origin of coordinates, is said to be a *neighborhood of the point at infinity*.

Find the limits in the following sequences (we give only the nth term in each sequence):

82. $z_n = \left(1 + \frac{1}{n}\right)e^{i\frac{\pi}{n}}$. 83. $z_n = \frac{i^n}{n}$.

84. $z_n = (1+3i)^n$. 85. $z_n = \frac{e^{in}}{n^2}$.

86. $z_n = \dfrac{n+2i}{3n+7i}$. 87. $z_n = e^{-i\left(\frac{\pi}{2}+\frac{1}{2n}\right)}$.

88. $z_n = n \sin \dfrac{i}{n}$. 89. $z_n = n \cos \dfrac{n\pi}{2} + in \sin \dfrac{n\pi}{2}$.

90. $z_n = (\sinh in)/n$.

91. Suppose we know that

$$\lim_{n\to\infty} z_n = \infty, \quad z_n = x_n + iy_n.$$

What can you say about the existence of limits for the sequences $\{x_n\}$ and $\{y_n\}$ as $n \to \infty$?

2. *Definition 3.* A *neighborhood of point* z_0 in the complex z plane is any domain that contains this point; the δ-*neighborhood* of point z_0 is the set of all points z lying inside a circle of radius δ centered at z_0, i.e. the set of all points z whose distance from z_0 is less than δ: $|z - z_0| < \delta$.

Let a function $f(z)$ be defined in a certain neighborhood Ω of point z_0 except, perhaps, at point z_0.

Definition 4. A number A is said to be the *limit of* $f(z)$ *at* z_0 if to each positive ε there corresponds a number $\delta = \delta(\varepsilon) > 0$ such that for all points $z \in \Omega$ that satisfy the condition $0 < |z - z_0| < \delta$ we have

$$|f(z) - A| < \varepsilon.$$

We then write

$$\lim_{z\to z_0} f(z) = A.$$

(Here we have assumed that z_0 and A are finite points in the complex z plane.)

Here is another definition of limit of a function at a point. Suppose a function $f(z)$ is defined inside an Ω-neighborhood of point z_0 except, perhaps, at point z_0.

Definition 5. If for each sequence $\{z_n\}$, $z_n \neq z_0$, that converges to z_0 the corresponding sequence of function values $\{f(z_n)\}$ converges to a unique complex number A, this number is said to be the *limit of* $f(z)$ *at point* z_0:

$$\lim_{z\to z_0} f(z) = A.$$

(Both z_0 and A may be infinite.)

The fact that there exists a limit of $f(z)$ as $z \to z_0$, where

$$f(z) = u(x, y) + iv(x, y), \quad z_0 = x_0 = iy_0,$$

is equivalent to the existence of two limits,

$$\lim_{\substack{x\to x_0\\ y\to y_0}} u(x, y) \quad \text{and} \quad \lim_{\substack{x\to x_0\\ y\to y_0}} v(x, y)$$

with

$$\lim_{z\to z_0} f(z) = \lim_{\substack{x\to x_0\\ y\to y_0}} u(x, y) + i \lim_{\substack{x\to x_0\\ y\to y_0}} v(x, y).$$

Limits of functions of a complex variable have the following properties. Suppose that

$$\lim_{z\to z_0} f(z) = A, \quad \lim_{z\to z_0} g(z) = B.$$

Then

$$\lim_{z\to z_0} [f(z) \pm g(z)] = A \pm B,$$

$$\lim_{z\to z_0} [f(z)\, g(z)] = AB,$$

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}, \quad B \neq 0.$$

Definition 6. A function $f(z)$ defined in a domain D is said to be *continuous at point* $z_0 \in D$ if

$$\lim_{z\to z_0} f(z) = f(z_0).$$

In other words, $f(z)$ is continuous at z_0 if for every positive ε we can select a positive $\delta = \delta(\varepsilon)$ such that for all points $z \in D$ satisfying the condition $|z - z_0| < \delta$ we have $|f(z) - f(z_0)| < \varepsilon$.

A continuous function of a complex variable

$$f(z) = u(x, y) + iv(x, y)$$

is continuous at point $z_0 = x_0 + iy_0$ if and only if its real and imaginary parts, $u(x, y)$ and $v(x, y)$, are continuous at point (x_0, y_0) in x and y simultaneously.

Definition 7. A function $f(z)$ of a complex variable z is said to be *continuous in a domain* D if it is continuous at each point of the domain.

The sum, the difference, and the product of two functions of a complex variable, $f(z)$ and $g(z)$, that are continuous in a domain D are also continuous functions in the same

domain, while the function $f(z)/g(z)$ is continuous at all points of D, where $g(z) \neq 0$.

If function $f(z)$ is continuous at point z_0 and the function $F(\tau)$ at point $\tau_0 = f(z_0)$, the composite function $F[f(z)]$ is continuous at z_0.

Example 6. Given the linear function

$$w = f(z) = az + b,$$

where a and b are complex-valued constants. Prove that at point z_0 this function has a limit equal to $w_0 = az_0 + b$, $a \neq 0$.

Proof. Indeed, let us take an arbitrary number $\varepsilon > 0$. Since

$$|f(z) - w_0| = |(az + b) - (az_0 + b)| = |az - az_0| = |a| \cdot |z - z_0|,$$

we obtain, by selecting a positive $\delta(\varepsilon)$ in such a way that $\delta = \varepsilon/|a|$, the following inequality: $|f(z) - w_0| < \varepsilon$ at $0 < |z - z_0| < \delta$. This means that $w_0 = az_0 + b$ is the limit of $f(z) = az + b$ at point z_0. But since

$$\lim_{z \to z_0} f(z) = az_0 + b = f(z_0),$$

we have proved that the linear function is continuous at all points in the complex plane.

Example 7. Show that $w = z^2$ is continuous for all values of z.

Solution. We take an arbitrary point z_0 and an arbitrary number $\varepsilon > 0$. Since the value of $f(z) = z^2$ at z_0 is $f(z_0) = z_0^2$, we can show that there is a positive $\delta(\varepsilon)$ such that $|z^2 - z_0^2| < \varepsilon$ for $|z - z_0| < \delta$. We send z to z_0. There is a positive M such that $|z| < M$ and $|z_0| < M$. Then

$$|z^2 - z_0^2| = |(z + z_0)(z - z_0)| = |z + z_0| \cdot |z - z_0| < (|z| + |z_0|)|z - z_0| < 2M|z - z_0|.$$

If we put $\delta = \varepsilon/2M$, it follows from $|z - z_0| < \delta$ that

$$|z^2 - z_0^2| < 2M\delta \leqslant \varepsilon,$$

i.e. the function $w = z^2$ is continuous at all points in the complex domain.

Using the definition of a limit, show that

92. $\lim\limits_{z\to 1}\dfrac{2z+1}{z+2}=1$. 93. $\lim\limits_{z\to 3-4i}|z|=5$.

Evaluate the following limits:

94. $\lim\limits_{z\to -i}\dfrac{z^2+3iz-2}{z+i}$.

95. $\lim\limits_{z\to \pi/4}\dfrac{\cos 2z}{\cosh iz+i\sinh iz}$.

96. $\lim\limits_{z\to 0}\dfrac{\sin z}{\sinh iz}$. 97. $\lim\limits_{z\to -i\pi/2}\dfrac{e^{2z}+1}{e^z+i}$.

Prove that the following functions are continuous in the entire complex plane:

98. $f(z)=\bar{z}$. 99. $f(z)=\operatorname{Re} z$. 100. $f(z)=\operatorname{Im} z$.

101. $P_n(z)=a_0z^n+a_1z^{n-1}+\ldots+a_n$, where the a_k ($k=0, 1, 2, \ldots, n$) are complex-valued constants.

102. Prove that a rational function $R(z)=P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials, is continuous at all points of the complex z plane where $Q(z)\neq 0$.

103. Show that the function $w=e^z$ is continuous in the entire complex plane.

§ 4 Differentiating Functions of a Complex Variable. The Cauchy-Riemann Conditions

Suppose we have a function $w=f(z)$ that is defined in a domain D in the complex z plane. Take two points, z and $z+\Delta z$, in D. We introduce the notations

$$\Delta w=f(z+\Delta z)-f(z),\quad \Delta z=\Delta x+i\,\Delta y.$$

Definition 1. A function $w=f(z)$ is said to be *differentiable at point* $z\in D$ if the ratio $\Delta w/\Delta z$ has a finite limit as Δz tends to zero in an arbitrary manner. This limit is called the *derivative of* $f(z)$ at the given point z and is denoted $f'(z)$ or w' or dw/dz, so that by the definition,

$$w'=f'(z)=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}. \tag{4.1}$$

If $z = x + iy$ and $w = f(z) = u(x, y) + iv(x, y)$, then at each point where $f(z)$ is differentiable we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{4.2}$$

which are known as the *Cauchy-Riemann conditions.*

Conversely, if at a point (x, y) the functions $u(x, y)$ and $v(x, y)$ are differentiable as functions of two real variables x and y and, moreover, satisfy the Cauchy-Riemann conditions (4.2), the function $f(z) = u + iv$ is differentiable at $z = x + iy$ as a function of the complex variable z.

Definition 2. A function $w = f(z)$ is said to be *analytic at a given point* $z \in D$ if it is differentiable at point z and in a neighborhood of this point. A function $f(z)$ is said to be *analytic in a domain* D if it is differentiable at each point of this domain. For an analytic function we always have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}. \tag{4.3}$$

Example 1. Show that the function $w = e^z$ is analytic in the entire complex plane.

Solution. We have $e^z = e^x(\cos y + i \sin y)$, so that

$$u(x, y) = e^x \cos y, \quad v(x, y) = e^x \sin y.$$

The functions $u(x, y)$ and $v(x, y)$, as functions of the real variables x and y, are differentiable at any point (x, y) (they have continuous partial derivatives of arbitrary high order) and satisfy the Cauchy-Riemann conditions (4.2). Consequently, $w = e^z$ is analytic in the entire complex plane. For $f(z) = e^z$, according to (4.3), we have

$$(e^z)' = (e^x \cos y)'_x + i(e^x \sin y)'_x = e^x(\cos y + i \sin y) = e^z.$$

Thus, $(e^z)' = e^z$.

Example 2. Is there at least one point in the complex domain where the function $w = \bar{z}z$?

Solution. Since $\bar{z}z = x^2 + y^2$, we see that

$$u(x, y) = x^2 + y^2, \quad v(x, y) \equiv 0.$$

The Cauchy-Riemann conditions in this case are

$$\begin{cases} 2x = 0 \\ 2y = 0 \end{cases}$$

and are satisfied only at point (0, 0). Hence, the function $w = z\bar{z}$ is differentiable only at $z = 0$ and is nowhere analytic.

Let us show, using definition 1, that $f(z) = z\bar{z}$ is differentiable at point $z = 0$. Indeed, since $f(0) = 0$, we find that

$$\Delta f = f(0 + \Delta z) - f(0) = \Delta z \cdot \overline{\Delta z}$$

and

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z \cdot \overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} (\Delta x - i\,\Delta y) = 0.$$

Thus, the derivative $f'(0)$ exists and is zero.

Example 3. Is the function $w = \bar{z} = x - iy$ analytic?

Solution. Here $u(x, y) = x$ and $v(x, y) = -y$ are everywhere differentiable functions of x and y. Moreover,

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = -1,$$

so that $\partial u/\partial x \neq \partial v/\partial y$, i.e. the first Cauchy-Riemann condition holds nowhere in the entire complex plane. This means that the function $w = \bar{z}$ is nowhere differentiable and, hence, nowhere analytic.

Using the Cauchy-Riemann conditions, find which of the following functions are analytic at least at one point.

104. (a) $w = z^2\bar{z}$; (b) $w = ze^z$; (c) $w = |z|\bar{z}$; (d) $w = e^{z^2}$; (e) $w = |z| \operatorname{Re} \bar{z}$; (f) $w = \sin 3z - i$.

105. (a) $w = \bar{z} \operatorname{Re} z$; (b) $w = \bar{z} \operatorname{Im} z$; (c) $w = |z| \operatorname{Im} z$; (d) $w = \cosh z$.

106. Show that in the domain $\operatorname{Re} z > 0$ the function $w = \ln z$ is analytic.

107. Show by direct calculations that

$$(z^n)' = nz^{n-1}$$

for any positive integer n.

108. Show that if the analytic functions $f(z)$ and $\varphi(z)$ satisfy the condition $f'(z) = \varphi'(z)$, then $f(z) = \varphi(z) + \text{const}$.

109. Show that in going over from Cartesian coordinates (x, y) to polar coordinates (ρ, φ) the Cauchy-Riemann conditions take the form

$$\frac{\partial u}{\partial \rho} = \frac{1}{\rho} \frac{\partial v}{\partial \varphi}, \quad \frac{\partial v}{\partial \rho} = -\frac{1}{\rho} \frac{\partial u}{\partial \varphi}. \tag{4.4}$$

110. Show that if an analytic function $w = f(z)$ is real-valued in a domain, it is constant.

111. Show that if $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain, then in this domain

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0.$$

112. Suppose that a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D. Show that family of curves $u(x, y) = \text{const}$ and $v(x, y) = \text{const}$ are orthogonal.

113. Show that the modulus and argument of the analytic function

$$f(z) = R(x, y)\, e^{i\Phi(x, y)}$$

are coupled through the following relations:

$$\frac{\partial R}{\partial x} = R\frac{\partial \Phi}{\partial y}, \quad \frac{\partial R}{\partial y} = -R\frac{\partial \Phi}{\partial x}.$$

Using the Cauchy-Riemann conditions, we can always construct an analytic function $f(z)$ if we know its real part $u(x, y)$ or imaginary part $v(x, y)$ (see *Sidorov et al.*). Moreover, a function $f(z)$ analytic in a neighborhood of a point z_0 can be reconstructed by one of the following formulas (see *Lavrent'ev and Shabat*):

$$f(z) = 2u\left(\frac{z+\bar{z}_0}{2}, \frac{z-\bar{z}_0}{2i}\right) - C_0, \tag{4.5}$$

$$f(z) = 2iv\left(\frac{z+\bar{z}_0}{2}, \frac{z-\bar{z}_0}{2i}\right) + C_0, \tag{4.6}$$

where $\bar{C}_0$ is the complex conjugate of $C_0 = f(z_0)$ and $\bar{z}_0$ the complex conjugate of z_0.

Example 4. Find the analytic function $w = f(z)$ from its known real part $u(x, y) = 2e^x \cos y$ and an additional condition that $f(0) = 2$.

Solution. We have $\partial u/\partial x = 2e^x \cos y$. The first Cauchy-Riemann condition implies that $\partial u/\partial x = \partial v/\partial y$, so that $\partial v/\partial y = 2e^x \cos y$. This means that $v(x, y) = \int 2e^x \cos y\, dy = 2e^x \sin y + \varphi(x)$, where $\varphi(x)$ is yet to be found. Finding the derivative of $v(x, y)$ with respect to x and using the second Cauchy-Riemann condition, we obtain

$$2e^x \sin y + \varphi'(x) = -\frac{\partial u}{\partial y} = 2e^x \sin y,$$

which implies that $\varphi'(x) = 0$ and, hence, $\varphi(x) = C$, where C is constant. Thus, $v(x, y) = 2e^x \sin y + C$ and therefore

$$f(z) = 2e^x \cos y + i(2e^x \sin y + C) = 2e^z + iC.$$

C can be found from the condition that $f(0) = 2$, i.e. $2e^0 + iC = 2$, or $C = 0$. The answer is $f(z) = 2e^z$.

Alternative solution. Let us use (4.5). We have $u(x, y) = 2e^x \cos y$, $z_0 = 0$, and $C_0 = 2$. This means that via (4.5) we have $f(z) = 2 \times 2e^{z/2} \cos(z/2i) - 2$. Since $\cos(z/2i) = \cos(-iz/2) = \cosh(z/2)$, we finally have $f(z) = 2e^z$.

Example 5. Find the analytic function $w = f(z)$ if its imaginary part is $v(x, y) = 3x + 2xy$ and if $f(-i) = 2$.

Solution. We use formula (4.6). In our case $v(x, y) = 3x + 2xy$, $z_0 = -i$, and $C_0 = 2$, so that

$$f(z) = 2i\left(3\frac{z+i}{2} + 2\frac{z+i}{2}\frac{z-i}{2i}\right) + 2 = 3iz + z^2.$$

Reconstruct the function $f(z)$ analytic in a neighborhood of z_0 from the known real part $u(x, y)$ or imaginary part $v(x, y)$ and the value $f(z_0)$:

114. (a) $u = \dfrac{x}{x^2+y^2}$, $f(\pi) = -\dfrac{1}{\pi}$;

(b) $v = \arctan(y/x)$ $(x > 0)$, $f(1) = 0$;

(c) $u = x^2 - y^2 + 2x$, $f(i) = 2i - 1$.

115. (a) $v = 2(\cosh x \sin y - xy)$, $f(0) = 0$;

(b) $u = 2 \sin x \cosh y - x$, $f(0) = 0$;

(c) $v = 2(2 \sinh x \sin y + xy)$, $f(0) = 3$.

116. (a) $v = -2 \sin 2x \sinh 2y + y$, $f(0) = 2$;

(b) $v = 2 \cos x \cosh y - x^2 + y^2$, $f(0) = 2$.

Definition 3. A function $\varphi(x, y)$ is said to be *harmonic in a domain D* if in this domain it has continuous partial derivatives up to the second order inclusive and satisfies Laplace's equation.

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0.$$

If a function $f(z) = u + iv$ is analytic in a domain D, both its real part $u(x, y)$ and imaginary part $v(x, y)$ are harmonic in D. However, if $u_1(x, y)$ and $v_1(x, y)$ are two harmonic functions, $f_1(z) = u_1(x, y) + iv_1(x, y)$ may not be analytic: the analyticity of $f_1(z)$ requires that u_1 and v_1 satisfy the Cauchy-Riemann conditions.

Two harmonic functions that satisfy the Cauchy-Riemann conditions are said to be *conjugate harmonic* (the ordering of the functions in the pair is essential).

117. Show that the following functions are harmonic:
(a) $u = x^2 + 2x - y^2$; (b) $u = 2e^x \cos y$; (c) $u = x/(x^2 + y^2)$; (d) $u = -y/(x^2 + y^2)$; (e) $u = \arctan (y/x)$; (f) $u = \ln (x^2 + y^2)$.

118. Can the following functions be the real or imaginary part of an analytic function $f(z) = u(x, y) + iv(x, y)$:
(a) $u = x^2 - y^2 + 2xy$; (b) $u = x^2$; (c) $v = \ln (x^2 + y^2)$; (d) $v = (x^2 + 1) y^2/2$.

119. When is the trinomial $u = ax^2 + 2bxy + cy^2$ a harmonic function?

The following problems contain pairs of harmonic functions, u and v. Find among them pairs of conjugate harmonic functions:

120. (a) $u = 3(x^2 - y^2)$, $v = 3x^2y - y^3$;

(b) $u = x/(x^2 + y^2)$, $v = -y/(x^2 + y^2)$;

(c) $u = x$, $v = -y$; (d) $u = e^x \cos y + 1$, $v = 1 + e^x \sin y$.

121. Suppose $u(x, y)$ is harmonic in a domain D, where it has continuous partial derivatives of all orders. Show that the derivatives are also harmonic in D.

122. Let $u = u(x, y)$ be harmonic in a domain D. Find all the functions f for which $f(u)$ is harmonic in D.

123. Let the function $w = f(z)$ be analytic in a domain D. Which of the functions
(a) $|w|$; (b) $\arg w$; (c) $\ln |w|$ are harmonic in D?

124. Prove that the product of functions $u(x, y)$ and $v(x, y)$ conjugate harmonic in a domain D is harmonic in D, too.

125. Suppose $u = u(x, y)$ and $v = v(x, y)$ form a pair of conjugate harmonic functions in a domain D. Which of the following pairs of functions form conjugate harmonic functions in D:
(a) $Au - Bv$, $Bu + Av$, with A and B constants;
(b) $u^2 - v^2$, uv; (c) $e^u \cos v$, $e^u \sin v$?

Example 6. Find all the harmonic functions of the type $u = f(x^2 + y^2)$ that are not constants.

Solution. Since the sought functions must be harmonic, they must satisfy Laplace's equation

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{4.7}$$

Substitute the given function into Eq. (4.7). To this end we will find its second-order derivatives. We put $t = x^2 + y^2$. We then have $u = f(t)$, with $t = t(x, y)$. The rule for finding the derivatives of a composite function yields

$$\frac{\partial u}{\partial x} = f'(t)\frac{\partial t}{\partial x}, \quad \frac{\partial u}{\partial y} = f'(t)\frac{\partial t}{\partial y},$$

$$\frac{\partial^2 u}{\partial x^2} = f''(t)\left(\frac{\partial t}{\partial x}\right)^2 + f'(t)\frac{\partial^2 t}{\partial x^2},$$

$$\frac{\partial^2 u}{\partial y^2} = f''(t)\left(\frac{\partial t}{\partial y}\right)^2 + f'(t)\frac{\partial^2 t}{\partial y^2}.$$

Adding the last two equations yields

$$\left[\left(\frac{\partial t^2}{\partial x}\right) + \left(\frac{\partial t}{\partial y}\right)^2\right] f''(t) - \left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2}\right) f'(t) = 0,$$

or

$$tf''(t) + f'(t) = 0.$$

We have thus arrived at *Euler's equation* for f,

$$t^2 f''(t) + tf'(t) = 0,$$

whose general solution is

$$f(t) = C_1 \ln t + C_2,$$

where C_1 and C_2 are constants. Thus, the sought harmonic functions have the form

$$u = f(x^2 + y^2) = C_1 \ln (x^2 + y^2) + C_2.$$

In the following problems find all the harmonic functions of the given form:

126. $x = f(ax + by)$, where a and b are constants.

127. $u = f(xy)$. **128.** $u = f(y/x)$.

129. $u = f(x^2 - y^2)$. **130.** $u = f(x + \sqrt{x^2 + y^2})$.

131. $u = f\left(\frac{x^2+y^2}{x}\right)$.

The geometric interpretation of the modulus and argument of a derivative

Suppose a function $f(z)$ is analytic at point z_0 and suppose that $f'(z_0) \neq 0$. Then $|f'(z_0)|$ is the modulus of expansion (or stretching) at point z_0 when the complex z plane is mapped into the complex w plane via $w = f(z)$, namely, when $|f'(z_0)| > 1$ we have stretching and when $|f'(z_0)| < 1$ shrinking.

The argument of $f'(z_0)$ is, geometrically speaking, the angle through which the tangent at point z_0 to any smooth curve, passing through z_0, in the complex z plane is rotated in the course of the transformation $w = f(z)$ so as to bring us to the tangent at point $w_0 = f(z_0)$ to the map of the smooth curve in the complex w plane. Note that if $\varphi = \arg f'(z) > 0$, the rotation is counterclockwise, while if $\varphi < 0$ it is clockwise.

Example 7. Find the modulus of expansion and the rotation angle in the transformation $w = z^2$ at point $z_0 = \sqrt{2} + i\sqrt{2}$.

Solution. We have $w'(z) = 2z$, so that $w'|_{z=\sqrt{2}+i\sqrt{2}} = 2\sqrt{2} + i2\sqrt{2}$. Going over from the algebraic form to the polar, we obtain

$$2\sqrt{2} + i2\sqrt{2} = 4\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) = 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

This means that

$$|f'(z)|_{z=\sqrt{2}+i\sqrt{2}} = 4, \quad \arg f'(z)|_{z=\sqrt{2}+i\sqrt{2}} = \frac{\pi}{4},$$

i.e. the modulus of expansion $r = 4$ (stretching), and the rotation angle $\varphi = \pi/4$ (counterclockwise).

Find the modulus of expansion r and the rotation angle φ for the given transformations $w = f(z)$ and at given points:

132. (a) $w = e^z$ at points $z_1 = \ln 2 + i\pi/4$ and $z_2 = -1 - i\pi/2$;

(b) $w = \sin z$ at points $z_1 = 0$ and $z_2 = 1 + i$;

(c) $w = z^3$ at points $z_1 = 2 - i$ and $z_2 = 1 + i\pi/2$.

133. Which part of the complex plane stretches and which shrinks under the following transformations:

(a) $w = e^z$; (b) $w = \ln z$; (c) $w = 1/z$; (d) $w = z^3$.

If a function $f(z)$ is analytic in a domain D and maps D into a domain $\tilde{D}$ in a one-to-one manner, the curve L lying in D is mapped into a curve $\tilde{L}$ in the complex w plane, and the length of this latter curve is

$$l_w = \int_L |f'(z)|\,|dz|. \qquad (4.8)$$

The domain D in the z plane under the mapping $w = f(z)$ becomes the domain $\tilde{D}$ in the w plane, and the area of $\tilde{D}$ is

$$S_{\tilde{D}} = \iint_D |f'(z)|^2\,dx\,dy. \qquad (4.9)$$

Thus, $|f'(z)|^2$ is the modulus (coefficient) of distortion of the area of D under the transformation $w = f(z)$.

Example 8. The point $z = x + iy$ traces the segment

$$x = 1, \quad -1 \leqslant y \leqslant 1. \qquad (4.10)$$

What is the length of the curve obtained as a result of mapping this segment via the function $w = z^2$?

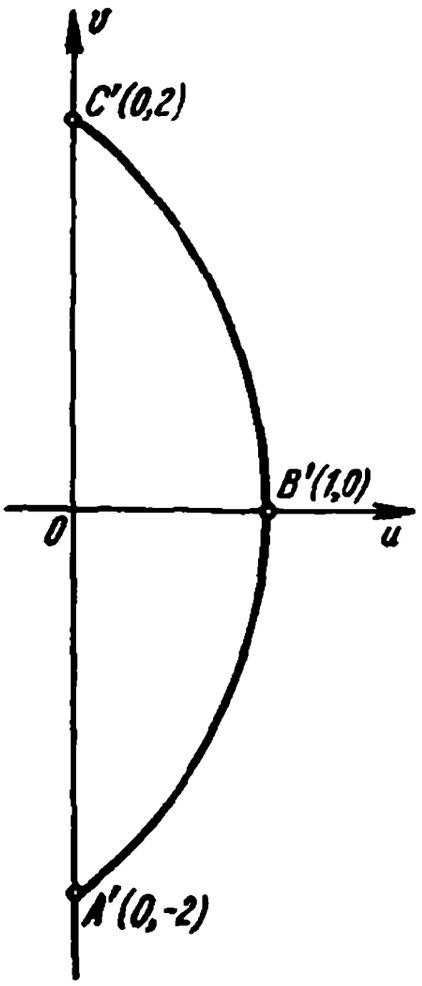


Figure 3

Solution. We have $w = z^2$, or

$$u + iv = x^2 - y^2 + 2ixy,$$

i.e.

$$\begin{cases} u = x^2 - y^2, \\ v = 2xy. \end{cases}$$

Obviously, on the segment given by (4.10) we have

$$\begin{cases} u = 1 - y^2, \\ v = 2y, \end{cases} \qquad (4.11)$$

and as y varies between -1 and $+1$, v will vary between -2 and $+2$. From (4.11) we then obtain the equation of the parabola (Fig. 3)

$$u = 1 - v^2/4. \qquad (4.12)$$

The length of the arc $A'B'C'$ of parabola (4.12) is

$$l_w = 2\int_0^2 \sqrt{1+\frac{v^2}{4}}\,dv = \int_0^2 \sqrt{4+v^2}\,dv$$

$$= 2\sqrt{2} + \ln(3+2\sqrt{2}).$$

Alternative solution. Using (4.8), we have

$$l_w = \int_L |f'(z)|\,|dz| = \int_L |2z|\,|dz| = 2\int_{-1}^{1} \sqrt{1-y^2}\,dy$$

$$= 4\int_0^1 \sqrt{1+y^2}\,dy = 2\sqrt{2} + \ln(3+2\sqrt{2}).$$

Example 9. Calculate the area of the domain into which the transformation $w = e^z$ maps the square

$$a-\varepsilon \leqslant x \leqslant a+\varepsilon, \quad -\varepsilon \leqslant y \leqslant \varepsilon \quad \text{(Fig. 4)}$$

(a is real, $0 < \varepsilon < \pi$, and $z = x + iy$).

Calculate the limit of the ratio of the areas of these domains as $\varepsilon \to 0$.

Solution. We have $w = e^z = e^{x+iy} = e^x e^{iy}$, or $w = \rho e^{i\varphi}$, where $\rho = e^x$ and $\varphi = y$. Thus, under the transformation

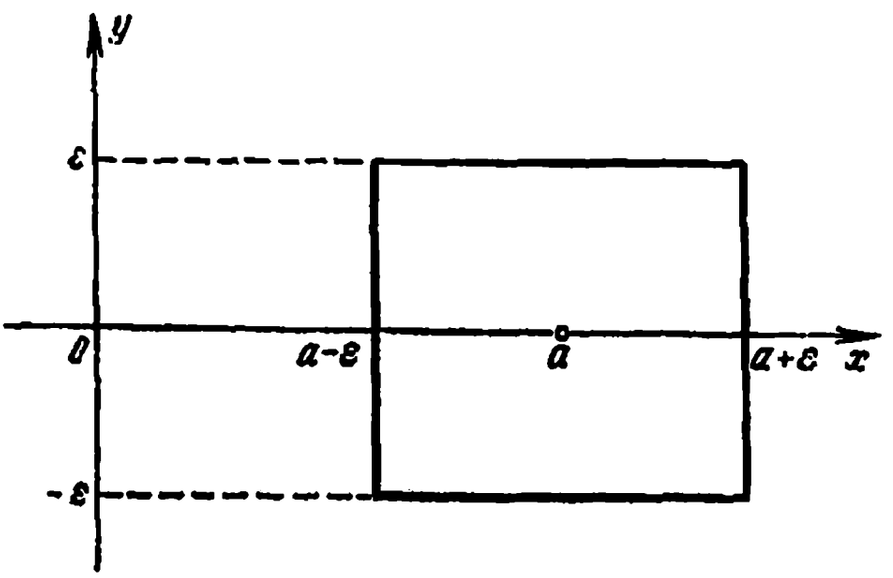


Figure 4

$w = e^z$ the square in the z plane becomes a domain in the w plane bounded by two rays, $\arg w = -\varepsilon$ and $\arg w = \varepsilon$, and the arcs inside these rays of two circles, $\rho = e^{a-\varepsilon}$ and

$\rho = e^{a+\varepsilon}$ (Fig. 5). The area of the resulting domain is

$$S_w = \int_{-\varepsilon}^{\varepsilon} d\varphi \int_{e^a-\varepsilon}^{e^a+\varepsilon} \rho\, d\rho = \varepsilon e^{2a-2\varepsilon}(e^{4\varepsilon}-1).$$

Alternative solution. Applying (4.9), we have

$$S_w = \iint_D |f'(z)|^2\, dx\, dy = \iint_D e^{2x}\, dx\, dy$$

$$= \int_{a-\varepsilon}^{a+\varepsilon} e^{2x}\, dx \int_{-\varepsilon}^{\varepsilon} dy = \varepsilon e^{2a-2\varepsilon}(e^{4\varepsilon}-1).$$

Obviously, the area S_z of D is equal to $4\varepsilon^2$ and whence

$$\lim_{\varepsilon\to 0} \frac{S_w}{S_z} = \lim_{\varepsilon\to 0} \frac{\varepsilon e^{2a-2\varepsilon}(e^{4\varepsilon}-1)}{4\varepsilon^2} = e^{2a}.$$

134. Find the area of the map of the square D $\{0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$ obtained as a result of the transformation $w = z^2$

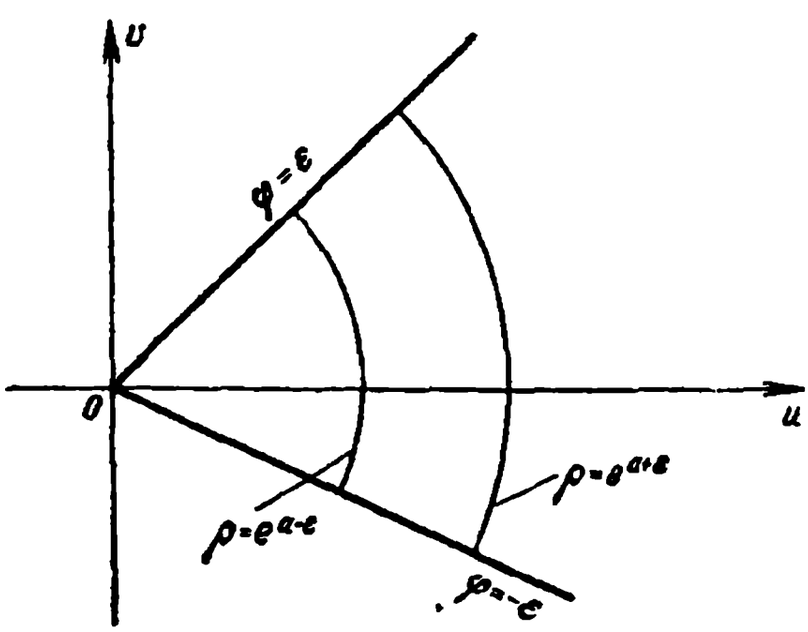


Figure 5

and the length of the boundary of the resulting domain.

135. Find the area of the map of the rectangle

$$P\left\{0 < x_1 \leqslant x \leqslant x_2 < \frac{\pi}{2},\ 0 < y_1 \leqslant y \leqslant y_2 < \frac{\pi}{2}\right\}$$

under the transformation $w = \cos z$.

136. Let z traverse a domain given by

$$1 \leqslant |z| \leqslant 2, \quad -\frac{\pi}{4} \leqslant \arg z \leqslant \frac{\pi}{4}.$$

Find the area of the domain obtained as a result of the transformation $w = z^2$.

137. Find the length L of the helix onto which the segment $y = x$, $0 \leqslant x \leqslant 2\pi$, is mapped via the function $w = e^z$.

138. Find the domain P_w into which the function $w = e^z$ maps the rectangle P $\{1 \leqslant x \leqslant 2, \ 0 \leqslant y \leqslant 8\}$. Calculate the area of P_w using (4.9) and explain why this formula yields an incorrect result.

139. Find the area of the figure obtained as a result of mapping the triangle bounded by the straight lines $x = 0$, $y = 0$, and $x + y = 1$ via the function $w = 1 + iz$.

§ 5 Integrating Functions of a Complex Variable

Suppose that a single-valued function $f(z)$ is defined and continuous in a domain D and that C is a piecewise smooth, closed or open oriented curve in D. Also suppose that

$$z = x + iy, \quad f(z) = u + iv,$$

where $u = u(x, y)$ and $v = v(x, y)$ are real-valued functions of the x and y variables.

Calculating the integral of a function $f(z)$ of complex variable z reduces to calculating ordinary line integrals, namely,

$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i \int_C v\,dx + u\,dy. \qquad (5.1)$$

The integral $\int_C f(z)\,dx$ generally depends on the path of integration C.

If $f(z)$ is analytic in a simply connected domain D, the value of the integral does not depend on the path of integration. In this case

$$\int_L f(z)\,dz = 0,$$

where L is any piecewise smooth, closed curve in D.

If curve C is given parametrically, i.e.

$$x = x(t), \quad y = y(t),$$

and the beginning and end of curve C correspond to $t = t_0$ and $t = t_1$, respectively, then

$$\int_C f(z)\,dz = \int_{t_0}^{t_1} [z(t)]\,z'(t)\,dt, \tag{5.2}$$

where

$$z(t) = x(t) + iy(t).$$

If the function $f(z)$ is analytic in a simply connected domain D containing points z_0 and z_1, the following Newton-Leibniz formula is valid:

$$\int_{z_0}^{z_1} f(z)\,dz = \Phi(z_1) - \Phi(z_0) = \Phi(z)\Big|_{z_0}^{z_1}, \tag{5.3}$$

where $\Phi(z)$ is an antiderivative of $f(z)$, i.e.

$$\Phi'(z) = f(z),$$

in D.

If two functions, say $f(z)$ and $\varphi(z)$, are analytic in a simply connected domain D and if z_0 and z_1 are any two points in D, then the following integration-by-parts formula is valid:

$$\int_{z_0}^{z_1} f(z)\,\varphi'(z)\,dz = [f(z)\,\varphi(z)]\Big|_{z_0}^{z_1} - \int_{z_0}^{z_1} \varphi(z)\,f'(z)\,dz.$$

Change of variables in integrals of functions of a complex variable is done in the same manner as for functions of a real variable. Suppose an analytic function $z = \varphi(w)$ maps in a one-to-one manner a contour C_1 in the complex w plane into a contour C in the z plane. Then

$$\int_C f(z)\,dz = \int_{C_1} f[\varphi(w)]\,\varphi'(w)\,dw.$$

If the path of integration is a ray starting at point z_0 or a circle centered at z_0, it is expedient to introduce the change of variables

$$z - z_0 = \rho e^{i\varphi}.$$

In the first case $\varphi = \text{const}$ and ρ is the real-valued integration variable, while in the second $\rho = \text{const}$ and φ is the real-valued integration variable.

Example 1. Evaluate the integral

$$\int_C (1+i-2\bar{z})\,dz$$

along the following curves connecting points $z_1 = 0$ and $z_2 = 1 + i$:

(1) the straight line connecting z_1 and z_2;

(2) the parabola $y = x^2$;

(3) the broken line $z_1z_2z_3$, with $z_3 = 1$.

Solution. We write the integrand in the form

$$1 + i - 2\bar{z} = (1 - 2x) + i\,(1 + 2y).$$

Here $u = 1 - 2x$ and $v = 1 + 2y$. Applying (5.1), we obtain

$$\int_C (1+i-2\bar{z})\,dz = \int_C (1-2x)\,dx - (1+2y)\,dy + i\int_C (1+2y)\,dx + (1-2x)\,dy.$$

(1) The equation of the straight line passing through the points $z_1 = 0$ and $z_2 = 1 + i$ is $y = x$, $0 \leqslant x \leqslant 1$, which means that $dy = dx$. Whence,

$$\int_C (1+i-2\bar{z})\,dz = \int_0^1 [(1-2x)-(1+2x)]\,dx + i\int_0^1 [(1+2x)+(1-2x)]\,dx = 2(i-1).$$

(2) For the parabola $y = x^2$ we have $dy = 2x\,dx$ $(0 \leqslant x \leqslant 1)$. Consequently,

$$\int_C (1+i-2\bar{z})\,dz = \int_0^1 [1-2x-(1+2x^2)\,2x]\,dx + i\int_0^1 [1+2x^2+(1-2x)\,2x]\,dx = -2+\frac{4}{3}\,i.$$

(3) On the segment z_1z_3 we have $y = 0$ and $dy = 0$, $0 \leqslant x \leqslant 1$. On the segment z_2z_3 we have $x = 1$ and $dx = 0$, $0 \leqslant y \leqslant 1$. Employing the linearity of line integrals, we

obtain

$$\int_C (1+i-2\bar{z})\,dz = \int_{z_1z_3} (1+i-2\bar{z})\,dz + \int_{z_3z_2} (1+i-2\bar{z})\,dz$$

$$= \int_0^1 (1-2x)\,dx + i\int_0^1 dx - \int_0^1 (1+2y)\,dy$$

$$+ i\int_0^1 (1-2\cdot 1)\,dy = -2.$$

The above example shows that the integral of a continuous but not analytic function generally depends on the shape of the integration path.

Example 2. Evaluate the integral

$$\int_C (z^2+z\bar{z})\,dz,$$

where C is an arc of the circle $|z| = 1$ $(0 \leqslant \arg z \leqslant \pi)$.

Solution. We put $z = e^{i\varphi}$. Then $dz = ie^{i\varphi}\,d\varphi$ and

$$\int_C (z^2+z\bar{z})\,dz = \int_0^\pi ie^{i\varphi}(e^{i2\varphi}+1)\,d\varphi$$

$$= i\int_0^\pi (e^{i3\varphi}+e^{i\varphi})\,d\varphi = \left(\frac{1}{3}e^{i3\varphi}+e^{i\varphi}\right)\Big|_0^\pi = -\frac{8}{3}.$$

Example 3. Evaluate the integral $\int_C e^{\bar{z}}\,dz$, where C is the segment of the straight line $y = -x$ connecting points $z_1 = 0$ and $z_2 = \pi - i\pi$.

Solution. The parametric equations of C are

$$x = t, \quad y = -t,$$

or, in complex form,

$$z = t - it,$$

where the real variable t varies between 0 and π. Applying (5.2), we obtain

$$\int_C e^{\bar{z}}\,dz = \int_0^\pi e^{t+it}(1-i)\,dt = (1-i)\int_0^\pi e^{(1+i)t}\,dt$$

$$= \frac{1-i}{1+i}e^{(1+i)t}\Big|_0^\pi = (e^\pi+1)\,i.$$

Example 4. Evaluate the integral

$$\int_{1-i}^{2+i} (3z^2 + 2z)\, dz.$$

Solution. Since the integrand $f(z) = 3z^2 + 2z$ is analytic everywhere, applying the Newton-Leibniz formula we find that

$$\begin{aligned}\int_{1-i}^{2+i} (3z^2 + 2z)\, dz &= (z^3 + z^2)\Big|_{1-i}^{2+i} \\ &= (2+i)^3 + (2+i)^2 - (1-i)^3 - (1-i)^2 \\ &= 7 + 19i.\end{aligned}$$

Example 5. Evaluate the integral

$$\int_0^i z \cos z\, dz.$$

Solution. The functions $f(z) = z$ and $\varphi(z) = \cos z$ are analytic in the entire complex z plane. Integrating by parts, we have

$$\begin{aligned}\int_0^i z \cos z\, dz &= \int_0^i z\, (\sin z)'\, dz = (z \sin z)\Big|_0^i - \int_0^i \sin z\, dz \\ &= i \sin i + \cos z\Big|_0^i \\ &= -\sinh 1 + \cosh 1 - 1 = (1-e)/e.\end{aligned}$$

Branches of multiple-valued functions and branch points

Suppose a function $w = f(z)$, analytic in a domain D, maps D into G and is such that the inverse, $z = \varphi(w)$, is a multiple-valued function in G. If in G there are single-valued analytic functions $z = \varphi_1(w)$, $z = \varphi_2(w)$, . . . for which the given function $w = f(z)$ is the inverse, then $\varphi_1(w)$, $\varphi_2(w)$, . . . are said to be *branches of* $\varphi(w)$ defined in G.

For example, the function $w = z^n$ for each point z_0 has exactly one point w_0 corresponding to z_0, but the function $z = \sqrt[n]{w}$ for each point w_0 ($w \neq 0$, $w \neq \infty$) has n different

points in the z plane; if $w = \rho e^{i\theta}$, then these n values of z are $z_k = re^{i\varphi_k}$, where

$$r = \sqrt[n]{\rho}, \quad \varphi_k = \frac{\theta}{n} + \frac{2k}{n}$$

$$(-\pi < \theta \leqslant \pi, \ k = 0, \ 1, \ 2, \ \ldots, \ n-1).$$

Suppose that the simply connected domain G contains the point w_0 but not the points $w = 0$ and $w = \infty$. Then different fixed values of k ($k = 0, 1, 2, \ldots, n-1$) for a fixed value of θ_0 (e.g. $\theta_0 = \arg w_0$) have corresponding to them different branches of the function $z = \sqrt[n]{w}$.

A point possessing the property that (one) full circle around it in a sufficiently small neighborhood changes one branch of a multiple-valued function to another branch of the same function is called a *branch point* of the function. For instance, points $w = 0$ and $w = \infty$ are the branch points of the function $\sqrt[n]{w}$. After the point $w = 0$ has been circled n times in the same direction we arrive at the initial branch of the function $\sqrt[n]{w}$; branch points possessing this property are said to be *algebraic* and having a *degree of ramification* $n - 1$, or *multiplicity* n. At each of such points the function under consideration has exactly one value, e.g. $\sqrt[n]{0} = 0$ and $\sqrt[n]{\infty} = \infty$; this means that at such points the various branches of the function merge.

For the logarithmic function $w = \operatorname{Ln} z$ the branch points are $z = 0$ and $z = \infty$, with $\operatorname{Ln} 0 = \infty$ and $\operatorname{Ln} \infty = \infty$. No finite number of circuits (in one direction) of the point $z = 0$ brings us to the initial branch of the function $\operatorname{Ln} z$. This is why such branch points are called *logarithmic*.

When integrating a multiple-valued function we must specify the branch of the function we are working with. This is done by specifying the value of this function at a certain point of the integration contour. If this contour, C, is closed, the point where the value of the integrand is specified is the initial point z_0 on the path of integration.

Example 6. Evaluate the integral

$$\int_C \frac{dz}{\sqrt{z}},$$

where C is the upper half of the circle $|z| = 1$. For $\sqrt{z}$ take the branch on which $\sqrt{1} = -1$.

Solution. The function $\sqrt{z}$ has two branches,

$$\sqrt{z} = \sqrt{|z|}\left(\cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2}\right),$$

$$\sqrt{z} = \sqrt{|z|}\left[\cos\left(\frac{\varphi}{2}+\pi\right) + i\sin\left(\frac{\varphi}{2}+\pi\right)\right]$$

$$= -|z|\left(\cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2}\right),$$

with $\varphi = \arg z$. Since the values of z are taken on the unit circle, $|z| = 1$ and, hence,

$$\sqrt{z} = \cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2}, \quad \sqrt{z} = -\cos\frac{\varphi}{2} - i\sin\frac{\varphi}{2}.$$

The condition $\sqrt{1} = -1$ is met on the second branch of $\sqrt{z}$,

$$\sqrt{z} = -\cos\frac{\varphi}{2} - i\sin\frac{\varphi}{2}. \tag{5.4}$$

Indeed, let $z = 1$. Then $\arg z = 0$ and

$$\sqrt{1} = -\cos 0 - i\sin 0 = -1.$$

Employing the Newton-Leibniz formula, we obtain

$$\int_C \frac{dz}{\sqrt{z}} = \int_1^{-1} \frac{dz}{\sqrt{z}} = 2\sqrt{z}\Big|_1^{-1} = 2(\sqrt{-1} - \sqrt{1}).$$

Putting $z = -1$ in (5.4), we find that

$$\sqrt{-1} = -\left[\cos\frac{\arg(-1)}{2} + i\sin\frac{\arg(-1)}{2}\right]$$

$$= -\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = -i.$$

Since by choice of branch $\sqrt{1} = -1$, we find that

$$\int_C \frac{dz}{\sqrt{z}} = 2(1-i).$$

Alternative solution. Put $z = \rho e^{i\varphi}$, with $\rho = 1$ and φ varying between 0 and π. From the condition $\sqrt{1} = -1$

it follows that $\sqrt{e^{i\varphi}} = e^{i(\varphi/2+\pi)}$. This yields

$$\int_C \frac{dz}{\sqrt{z}} = \int_0^{\pi} \frac{ie^{i\varphi}}{\sqrt{e^{i\varphi}}}\,d\varphi = \int_0^{\pi} \frac{ie^{i\varphi}\,d\varphi}{e^{i\left(\frac{\varphi}{2}+\pi\right)}} = \int_0^{\pi} ie^{i\left(\frac{\varphi}{2}-\pi\right)}\,d\varphi$$

$$= 2e^{i\left(\frac{\varphi}{2}-\pi\right)}\Big|_0^{\pi} = 2\left(e^{-i\frac{\pi}{2}} - e^{-i\pi}\right) = 2(1-i).$$

Example 7. Evaluate the integral $I = \int_1^i (\ln^3 z/z)\,dz$ along the circle $|z| = 1$ ($\ln z$ is the principal value of the logarithm, and $\ln 1 = 0$).

First solution. Employing the Newton-Leibniz formula yields

$$I = \int_1^i \frac{\ln^3 z}{z}\,dz = \int_1^i \ln^3 z\,d(\ln z) = \frac{\ln^4 z}{4}\Big|_1^i$$

$$= \frac{\ln^4 i - \ln^4 1}{4} = \frac{\ln^4 i}{4} = \frac{1}{4}\left(\frac{\pi i}{2}\right)^4 = \frac{\pi^4}{64}.$$

Second solution. We introduce the change of variable

$$\ln z = w, \quad dw = \frac{dz}{z}.$$

The chosen arc of the circle $|z| = 1$ is transformed to the segment on the imaginary axis between points (0, 0) and $(0, \pi/2)$. The integral is

$$I = \int_0^{\frac{\pi}{2} i} w^3\,dw = \frac{w^4}{4}\Big|_0^{\frac{\pi}{2} i} = \frac{1}{4}\,\frac{\pi^4 i^4}{2^4} = \frac{\pi^4}{64}.$$

Third solution. We put $z = e^{i\varphi}$ (here $\rho = |z| = 1$). Then

$$\ln z = i\varphi, \quad dz = ie^{i\varphi}\,d\varphi.$$

The real variable φ varies between 0 and $\pi/2$. We then have

$$I = \int_0^{\frac{\pi}{2}} \frac{i^3\varphi^3 e^{i\varphi} i\,d\varphi}{e^{i\varphi}} = \int_0^{\frac{\pi}{2}} \varphi^3\,d\varphi = \frac{\varphi^4}{4}\Big|_0^{\frac{\pi}{2}} = \frac{\pi^4}{64}.$$

Evaluate the following integrals:

140. $\int_C z \operatorname{Im} z^2\,dz$, C: $|z| = 1$ $(-\pi \leqslant \arg z \leqslant 0)$.

141. $\int_C e^{|z|^2} \operatorname{Re} z\, dz$, C is the straight line connecting points $z_1 = 0$ and $z_2 = 1 + i$.

142. $\int_C \ln z\, dz$ ($\ln z$ is the principal value of the logarithm), C: $|z| = 1$, and the initial point of the integration path is (a) $z_0 = 1$ and (b) $z_0 = -1$. The sense of circuit is counterclockwise.

143. $\int_C z \operatorname{Re} z\, dz$, C: $|z| = 1$. The sense of circuit is counterclockwise.

144. $\int_C z\bar{z}\, dz$, C: $|z| = 1$. The sense of circuit is counterclockwise.

145. $\int_1^i ze^z\, dz$.

146. $\int_C \operatorname{Re} z\, dz$, C: (a) $z = (2 + i)t$ $(0 \leqslant t \leqslant 1)$; (b) a broken line consisting of the segment $[0, 2]$ of the real axis and the segment connecting points $z_1 = 2$ and $z_2 = 2 + i$.

147. $\int_{1+i}^{-1-i} (2z+1)\, dz$. **148.** $\int_0^{i+1} z^3\, dz$. **149.** $\int_1^i (3z^4 - 2z^3)\, dz$.

150. $\int_C e^z\, dz$, C: (a) the arc of the parabola $y = x^2$ connecting points $z_1 = 0$ and $z_2 = 1 + i$; (b) the segment of the straight line connecting these points.

151. $\int_C \cos z\, dz$, C: the segment of the straight line connecting points $z_1 = \pi/2$ and $z_2 = \pi + i$.

152. $\int_C dz/\sqrt{z}$, C: (a) the upper half of the circle $|z| = 1$, and the selected branch of $\sqrt{z}$ is the one on which $\sqrt{1} = 1$; (b) $|z| = 1$, $\operatorname{Re} z \geqslant 0$, and the selected branch is the one on which $\sqrt{-i} = \sqrt{2}(1 - i)/2$.

153. $\int\limits_C dz/\sqrt[4]{z^3}$, C: the upper half of the circle $|z| = 1$, and the selected branch of $w = \sqrt[4]{z^3}$ is the one on which $\sqrt[4]{1} = 1$.

154. $\int\limits_{1+i}^{2i} (z^3 - z)\, e^{\frac{z^2}{2}}\, dz$. 155. $\int\limits_0^i z \cos z\, dz$.

156. $\int\limits_1^i z \sin z\, dz$. 157. $\int\limits_0^i (z - i)\, e^{-z}\, dz$.

158. $\int\limits_1^i \frac{\ln(z+1)}{z+1}\, dz$ along the circumference of $|z| = 1$ with $\operatorname{Im} z \geqslant 0$ and $\operatorname{Re} z \geqslant 0$.

159. $\int\limits_1^i \frac{\ln z}{z}\, dz$ along the segment of the straight line connecting points $z_1 = 1$ and $z_2 = i$.

160. $\int\limits_0^{1+i} \sin z \cos z\, dz$.

161. $\int\limits_1^i \frac{1 + \tan z}{\cos^2 z}\, dz$ along the straight line connecting points $z_1 = 1$ and $z_2 = i$.

162. $\int\limits_{-1}^i \frac{\cos z\, dz}{\sqrt{\sin z}}$ along the straight line connecting points $z_1 = -1$ and $z_2 = i$; the selected branch of $\sqrt{\sin z}$ is the one on which $\sqrt{\sin(-1)} = i\sqrt{\sin 1}$.

163. $\int\limits_C \operatorname{Re}(\sin z) \cos z\, dz$, C: $|\operatorname{Im} z| \leqslant 1$, $\operatorname{Re} z = \frac{\pi}{4}$.

164. $\int\limits_C z \operatorname{Im}(z^2)\, dz$, C: $|\operatorname{Im} z| \leqslant 1$, $\operatorname{Re} z = 1$.

165. $\int_{-i}^{i} ze^{z^2}\,dz$.

166. $\int_C \tan z\,dz$, C: the arc of parabola $y = x^2$ connecting points $z = 0$ and $z = 1 + i$.

§ 6 Cauchy's Integral Formula

If a function $f(z)$ is analytic in a domain D that has a piecewise smooth, closed contour C as boundary and on the boundary, *Cauchy's integral formula* is valid:

$$f(z_0) \quad \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0} \quad (z_0 \in D), \tag{6.1}$$

where C is traversed in such a way that D is always to the left. This formula enables evaluating several integrals.

Example 1. Evaluate the integral

$$\int_{|z|=2} \frac{\cosh iz}{z^2 + 4z + 3}\,dz.$$

Solution. Inside the circle $|z| = 2$ the denominator vanishes at point $z_0 = -1$. To employ (6.1) we rewrite the integral in the following manner:

$$\int_{|z|=2} \frac{\cosh iz}{z^2 + 4z + 3}\,dz = \int_{|z|=2} \frac{\cosh iz}{(z+1)(z+3)}\,dz = \int_{|z|=2} \frac{\dfrac{\cosh iz}{z+3}}{z-(-1)}\,dz.$$

Here $z_0 = -1$, and the function $f(z) = (\cosh iz)/(z + 3)$ is analytic in the circle $|z| \leqslant 2$. Whence,

$$\int_{|z|=2} \frac{\cosh iz}{z^2 + 4z + 3}\,dz = 2\pi i f(-1) = \frac{\cosh(-i)}{2}$$

$$= \pi i \cosh i = \pi i \cos 1.$$

Example 2. Employing Cauchy's integral formula, evaluate the integral $\int_C \frac{e^{z^2}}{z^2 - 6z}\,dz$ with

(1) C: $|z - 2| = 1$; (2) C: $|z - 2| = 3$; (3) C: $|z - 2| = 5$.

Solution. (1) In the domain bounded by the circle $|z-2|=1$ the integrand is an analytic function, whence in view of Cauchy's theorem

$$\int\limits_{|z-2|=1} \frac{e^{z^2}}{z^2-6z}\,dz = 0.$$

(2) In the domain bounded by the circle $|z-2|=3$ there is only one point at which the denominator vanishes, $z=0$. We write the integral in the form

$$\int\limits_C \frac{e^{z^2}}{z^2-6z}\,dz = \int\limits_{|z-2|=3} \frac{\frac{e^{z^2}}{z-6}}{z}\,dz.$$

The function $f(z) = e^{z^2}/(z-6)$ is analytic in the given domain. Employing Cauchy's integral formula ($z_0 = 0$), we obtain

$$\int\limits_{|z-2|=3} \frac{e^{z^2}}{z^2-6z}\,dz = 2\pi i \left.\frac{e^{z^2}}{z-6}\right|_{z=0} = 2\pi i\left(-\frac{1}{6}\right) = -\frac{\pi i}{3}.$$

(3) There are two points, $z=0$ and $z=6$, where the denominator of the integrand vanishes in the domain bounded by the circle $|z-2|=5$. Here (6.1) cannot be used directly, and so we proceed as follows.

First method. We expand $(z^2-6z)^{-1}$ in partial fractions:

$$\frac{1}{z^2-6z} = \frac{1}{6}\cdot\frac{1}{z-6} - \frac{1}{6}\cdot\frac{1}{z}.$$

Substitution into the integral yields

$$\int\limits_{|z-2|=5} \frac{e^{z^2}}{z^2-6z}\,dz = \frac{1}{6}\int\limits_{|z-2|=5} \frac{e^{z^2}\,dz}{z-6} - \frac{1}{6}\int\limits_{|z-2|=5} \frac{e^{z^2}\,dz}{z}$$

$$= \frac{1}{6}\,2\pi i e^{36} - \frac{1}{6}\,2\pi i = \frac{e^{36}-1}{3}\,\pi i.$$

Second method. We construct small circles γ_1 and γ_2 centered at points $z=0$ and $z=6$, i.e. so small that they do not intersect and lie entirely in the circle $|z-2| \leqslant 5$ (Fig. 6). In the triply connected domain bounded by circles $|z-2|=5$, γ_1, and γ_2 the integrand is everywhere analytic. By Cauchy's theorem for a multiply connected

domain,

$$\int\limits_{|z-2|=5} \frac{e^{z^2}dz}{z^2-6z} = \int\limits_{\gamma_1} \frac{e^{z^2}dz}{z^2-6z} + \int\limits_{\gamma_2} \frac{e^{z^2}dz}{z^2-6z}.$$

We can apply Cauchy's integral formula (6.1) to each integ-

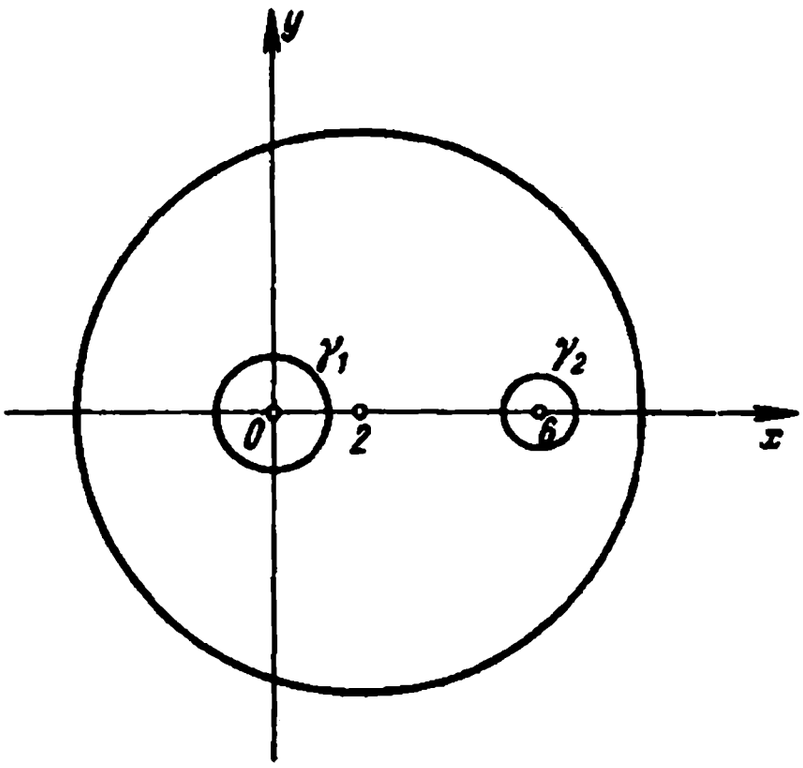


Figure 6

ral on the right-hand side. This yields

$$\int\limits_{|z-2|=5} \frac{e^{z^2}dz}{z^2-6z} = 2\pi i \frac{e^{z^2}}{z-6}\Big|_{z=0} + 2\pi i \frac{e^{z^2}}{z}\Big|_{z=6} = \frac{e^{36}-1}{3}\pi i.$$

Evaluate the following integrals by employing Cauchy's integral formula (all circles are circuited counterclockwise):

167. $\int\limits_{|z|=1} \frac{e^z}{z^2+2z}dz.$

168. $\int\limits_{|z-i|=1} \frac{e^{iz}}{z^2+1}dz.$

169. $\int\limits_{|z-1|=2} \frac{\sin(\pi z/2)}{z^2+2z-3}dz.$

170. $\int\limits_{|z|=2} \frac{\sin iz}{z^2-4z+3}\,dz.$

171. $\int\limits_{|z|=1} \frac{\tan z}{ze^{1/(z+2)}}\,dz.$

172. $\int\limits_{|z|=3} \frac{\cos(z+\pi i)}{z(e^z+2)}\,dz.$

173. $\int\limits_{|z|=5} \frac{dz}{z^2+16}.$

174. $\int\limits_{|z|=4} \frac{dz}{(z^2+9)(z+9)}.$

175. $\int\limits_{|z|=1} \frac{\sinh\frac{\pi}{2}(z+i)}{z^2-2z}.$

176. $\int\limits_{|z|=2} \frac{\sin z \sin(z-1)}{z^2-z}\,dz.$

If a function $f(z)$ is analytic in a domain D and on the domain's boundary C, then for all positive integers n

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int\limits_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}, \tag{6.2}$$

where $z_0 \in D$ and $z \in C$. This formula can be used to evaluate integrals.

Example 3. Evaluate the integral

$$\int\limits_{|z-1|=1} \frac{\sin \pi z}{(z^2-1)^2}\,dz.$$

Solution. The integrand $(\sin \pi z)(z^2-1)^{-2}$ is analytic in the domain $|z-1| \leqslant 1$ everywhere except at $z_0 = 1$. In the integrand we isolate a function $f(z)$ that is analytic in the circle $|z-1| \leqslant 1$. To this end we write the integrand in the form

$$\frac{\sin \pi z}{(z^2-1)^2} = \frac{\dfrac{\sin \pi z}{(z+1)^2}}{(z-1)^2},$$

and take $(\sin \pi z)(z+1)^{-2}$ as $f(z)$. Putting $n = 1$ in (6.2), we obtain

$$\int_{|z-1|=1} \frac{\frac{\sin \pi z}{(z+1)^2}}{(z-1)^2}\, dz = 2\pi i f'(1).$$

Next we find the derivative

$$f'(z) = \left(\frac{\sin \pi z}{(z+1)^2}\right)' = \frac{\pi \cos \pi z \cdot (z+1) - 2 \sin \pi z}{(z+1)^3}.$$

Whence,

$$f'(1) = \frac{2\pi \cos \pi}{2^3} = -\frac{\pi}{4}.$$

Consequently

$$\int_{|z-1|=1} \frac{\sin \pi z}{(z^2-1)^2}\, dz = -\frac{\pi^2}{2}\, i.$$

Example 4. Evaluate the integral

$$\int_{|z|=2} \frac{\cosh z\, dz}{(z+1)^3 (z-1)}.$$

Solution. The denominator $(z+1)^3(z-1)$ of the integrand vanishes at two points, $z_1 = -1$ and $z_2 = 1$, which lie in the circle $|z| \leqslant 2$. We proceed with the following partial fraction expansion:

$$\frac{1}{(z+1)^3(z-1)} = \frac{1}{8}\,\frac{1}{z-1} - \frac{1}{8}\,\frac{1}{z+1} - \frac{1}{4}\,\frac{1}{(z+1)^2} - \frac{1}{2}\,\frac{1}{(z+1)^3}.$$

Employing the linearity of the integral, we find that

$$\int_{|z|=2} \frac{\cosh z}{(z+1)^3(z-1)}\, dz = \frac{1}{8}\int_{|z|=2} \frac{\cosh z\, dz}{z-1} - \frac{1}{8}\int_{|z|=2} \frac{\cosh z}{z+1}\, dz$$
$$- \frac{1}{4}\int_{|z|=2} \frac{\cosh z}{(z+1)^2}\, dz - \frac{1}{2}\int_{|z|=2} \frac{\cosh z}{(z+1)^3}\, dz.$$

We apply Cauchy's integral formula (6.1) to the first and second integrals on the right-hand side. This yields

$$\int_{|z|=2} \frac{\cosh z}{z-1}\, dz = 2\pi i \cosh 1,$$

$$\int_{|z|=2} \frac{\cosh z}{z+1}\, dz = 2\pi i \cosh 1.$$

The third and fourth integrals are calculated via (6.2). This yields

$$\int\limits_{|z|=2} \frac{\cosh z}{(z+1)^2}\,dz = 2\pi i\,(\cosh z)'\Big|_{z=-1} = -2\pi i \sinh 1.$$

$$\int\limits_{|z|=2} \frac{\cosh z}{(z+1)^3}\,dz = \frac{2\pi i}{2!}\,(\cosh z)''\Big|_{z=-1} = \pi i \cosh 1.$$

We finally obtain

$$\int\limits_{|z|=2} \frac{\cosh z\,dz}{(z+1)^3\,(z-1)} = \frac{2\pi i \cosh 1}{8} - \frac{2\pi i \cosh 1}{8}$$

$$+ \frac{1}{4}\,2\pi i \sinh 1 - \frac{1}{2}\,\pi i \cosh 1$$

$$= \frac{\sinh 1 - \cosh 1}{2}\,\pi i = -\frac{\pi i}{2e}.$$

Alternative solution. We construct small circles γ_1 and γ_2 centered at $z_1 = -1$ and $z_2 = 1$, i.e. so small that they do not intersect and lie entirely within the circle $|z| \leqslant 2$. In the triply connected domain bounded by the circles $|z| = 2$, γ_1, and γ_2 the integrand is everywhere analytic. By Cauchy's theorem for a multiply connected domain,

$$\int\limits_{|z|=2} \frac{\cosh z\,dz}{(z+1)^3\,(z-1)} = \int\limits_{\gamma_1} \frac{\cosh z\,dz}{(z+1)^3\,(z-1)} + \int\limits_{\gamma_2} \frac{\cosh z\,dz}{(z+1)^3\,(z-1)}. \quad (6.3)$$

We apply (6.2) to the first integral on the right-hand side of (6.3), but before we write the integrand as

$$\frac{\cosh z}{(z+1)^3\,(z-1)} = \frac{\dfrac{\cosh z}{z-1}}{(z+1)^3}.$$

The function $(\cosh z)(z-1)^{-1}$ is analytic inside γ_1, so that, in view of (6.2),

$$\int\limits_{\gamma_1} \frac{\cosh z\,dz}{(z+1)^3\,(z-1)} = \int\limits_{\gamma_1} \frac{\dfrac{\cosh z}{z-1}}{(z+1)^3}\,dz$$

$$= \frac{2\pi i}{2!}\left(\frac{\cosh z}{z-1}\right)''\Big|_{z=-1} = -\frac{2e^{-1} + \cosh 1}{4}\,\pi i.$$

We apply Cauchy's integral formula (6.1) to the second integral on the right-hand side of (6.3). This yields

$$\int_{\gamma_2} \frac{\cosh z\, dz}{(z+1)^3 (z-1)} = \int_{\gamma_2} \frac{\dfrac{\cosh z}{(z+1)^3}}{z-1}\, dz = 2\pi i \left.\frac{\cosh z}{(z+1)^3}\right|_{z=1} = \pi i \frac{\cosh 1}{4}.$$

Finally we have

$$\int_{|z|=2} \frac{\cosh z\, dz}{(z+1)^3 (z-1)} = -\pi i \frac{2e^{-1}+\cosh 1}{4} + \pi i \frac{\cosh 1}{4} = -\frac{\pi i}{2e}.$$

Evaluate the following integrals:

177. $\displaystyle\int_{|z|=1} \frac{\cos^2 z}{z^3}\, dz.$ **178.** $\displaystyle\int_{|z|=1} \frac{\sinh^2 z}{z^3}\, dz.$

179. $\displaystyle\int_{|z-1|=1} \frac{\sin \frac{\pi}{4} z}{(z-1)^2 (z-3)}\, dz.$

180. $\displaystyle\int_{|z|=2} \frac{z \sinh z}{(z^2-1)^2}\, dz.$

181. $\displaystyle\int_{|z-3|=6} \frac{z\, dz}{(z-2)^3 (z+4)}.$

182. $\displaystyle\int_{|z-2|=3} \frac{\cosh e^{i\pi z}}{z^3-4z^2}\, dz.$

183. $\displaystyle\int_{|z|=1/2} \frac{1}{z^3} \cos \frac{\pi}{z+1}\, dz.$

184. $\displaystyle\int_{|z-2|=1} \frac{e^{1/z}}{(z^2+4)^2}\, dz.$

185. $\displaystyle\int_{|z|=1/2} \frac{1-\sin z}{z^2}\, dz.$

186. $\displaystyle\int_{|z-1|=1/2} \frac{e^{iz}}{(z^2-1)^2}\, dz.$

§ 7 Series in the Complex Domain

Suppose we have a series with complex-valued terms:

$$z_1 + z_2 + \ldots + z_n + \ldots = \sum_{n=1}^{\infty} z_n, \tag{7.1}$$

where $z_n = x_n + iy_n$. The series (7.1) is said to converge if and only if both

$$x_1 + x_2 + \ldots + x_n + \ldots = \sum_{n=1}^{\infty} x_n \tag{7.2}$$

and

$$y_1 + y_2 + \ldots + y_n + \ldots = \sum_{n=1}^{\infty} y_n. \tag{7.3}$$

converge. The series (7.1) is said to be absolutely convergent if

$$|z_1| + |z_2| + \ldots + |z_n| + \ldots = \sum_{n=1}^{\infty} |z_n|. \tag{7.4}$$

The series (7.2), (7.3), and (7.4) are series with real terms, and their convergence is studied by applying the known convergence tests in the real domain.

Example 1. Test the convergence of

$$\sum_{n=1}^{\infty} \frac{e^{in}}{n^2}.$$

Solution. We have $e^{in} = \cos n + i \sin n$, and so the convergence of the above series is reduced to the convergence of the two series with real terms,

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{\sin n}{n^2}.$$

Each series is absolutely convergent. Consequently, the series in question is absolutely convergent, too.

Example 2. Test the convergence of

$$\sum_{n=1}^{\infty} \frac{e^{i\frac{\pi}{n}}}{n}.$$

Solution. We have

$$e^{i\frac{\pi}{n}} = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}.$$

The series $\sum_{n=1}^{\infty}\frac{\cos(\pi/n)}{n}$ is divergent, while $\sum_{n=1}^{\infty}\frac{\sin(\pi/n)}{n}$ is convergent. Hence, the given series is divergent.

Test the following series for convergence:

187. $\sum_{n=1}^{\infty}\frac{\cos in}{2^n}$. 188. $\sum_{n=1}^{\infty}\frac{n\sin in}{3^n}$.

189. $\sum_{n=1}^{\infty}\frac{\cos in^2}{5n^2}$. 190. $\sum_{n=1}^{\infty}\frac{e^{i2n}}{n\sqrt{n}}$.

191. $\sum_{n=1}^{\infty}\frac{e^{i\frac{\pi}{n}}}{\sqrt{n}}$. 192. $\sum_{n=1}^{\infty}\frac{(1+i)^n}{2^{\frac{n}{2}}\cos in}$.

193. $\sum_{n=1}^{\infty}\frac{\sinh i\sqrt{n}}{\sin in}$. 194. $\sum_{n=1}^{\infty}\frac{\ln n}{\sinh in}$.

195. $\sum_{n=1}^{\infty}\frac{\cosh(i\pi/n)}{n^{\ln n}}$. 196. $\sum_{n=1}^{\infty}\frac{n}{\tan i\pi n}$.

Power Series

A series of the type

$$c_0 = c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots = \sum_{n=0}^{\infty} c_n z^n, \qquad (7.5)$$

where $c_0, c_1, \ldots$ are complex constants and z is the complex variable, is called a *power series* in the complex domain.

Abel's theorem *If the power series* (7.5) *converges for some value of* z *equal to* z_0, *it is (absolutely) convergent for all values of* z *such that* $|z| < |z_0|$. *If the series* (7.5) *diverges at* $z = z_1$, *then it is divergent for any value of* z *such that* $|z| > |z_1|$.

The domain of convergence of (7.5) is a circle centered at origin of coordinates, which is called the *circle of convergence of* (7.5). The *radius of convergence* R of a power series can be found thus:

$$R = \lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} \quad (c_n \neq 0) \tag{7.6}$$

or

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|c_n|}}, \tag{7.7}$$

provided that the limits exist.

Example 3. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \cos in \cdot z^n.$$

Solution. We have

$$c_n = \cos in = \frac{e^{-n}+e^n}{2} = \cosh n.$$

To find the radius of convergence we employ (7.6):

$$R = \lim_{n\to\infty} \frac{|\cosh n|}{|\cosh (n+1)|} = \lim_{n\to\infty} \frac{\cosh n}{\cosh (n+1)}$$

$$= \lim_{n\to\infty} \frac{\cosh n}{\cosh n \cosh 1 + \sinh n \sinh 1}$$

$$= \lim_{n\to\infty} \frac{1}{\cosh 1 + \tanh n \sinh 1} = \frac{1}{\cosh 1 + \sinh 1} = e^{-1}$$

since

$$\lim_{n\to\infty} \tanh n = \lim_{n\to\infty} \frac{e^n+e^{-n}}{e^n-e^{-n}} = \lim_{n\to\infty} \frac{1+e^{-2n}}{1-e^{-2n}} = 1.$$

Thus, the radius of convergence of the given power series is e^{-1}.

Example 4. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (1+i)^n z^n.$$

Solution. We will find the modulus of the coefficient $c_n = (1 + i)^n$:

$$|c_n| = |(1 + i)^n| = |1 + i|^n = (\sqrt{2})^n = 2^{n/2}.$$

Employing (7.7), we find the radius of convergence of the given series:

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{2^{n/2}}} = \frac{1}{\sqrt{2}}.$$

Find the radius of convergence of each of the following series:

197. $\sum_{n=1}^{\infty} e^{in} z^n$. 198. $\sum_{n=1}^{\infty} e^{i\frac{\pi}{n}} z^n$.

199. $\sum_{n=0}^{\infty} \left(\frac{z}{1-i}\right)^n$. 200. $\sum_{n=1}^{\infty} \left(\frac{z}{in}\right)^n$.

201. $\sum_{n=1}^{\infty} \cosh(iz^n/n)$. 202. $\sum_{n=1}^{\infty} \left(\frac{z}{\ln in}\right)^n$.

203. $\sum_{n=0}^{\infty} i^n z^n$. 204. $\sum_{n=1}^{n} \sin\frac{\pi i}{n} z^n$.

205. $\sum_{n=1}^{\infty} \cos^n \frac{\pi i}{\sqrt{n}} z^n$.

206. $\sum_{n=1}^{\infty} \frac{z^n}{\sin^n (1+in)}$.

207. $\sum_{n=0}^{\infty} (n+i) z^n$. 208. $\sum_{n=0}^{\infty} \cos in \cdot z^n$.

209. Suppose the series $\sum_{n=0}^{\infty} c_n z^n$ has a radius of convergence r and the series $\sum_{n=0}^{\infty} c'_n z^n$ a radius of convergence r'. Estimate the radius of convergence R of each of the

following series:

$$\text{(a)}\ \sum_{n=0}^{\infty}(c_n+c_n')\,z^n;\quad \text{(b)}\ \sum_{n=0}^{\infty}(c_n-c_n')\,z^n;\quad \text{(c)}\ \sum_{n=0}^{\infty}c_nc_n'z^n;$$

$$\text{(d)}\ \sum_{n=0}^{\infty}\frac{c_n}{c_n'}\,z^n\quad (c_n'\neq 0).$$

Taylor and Laurent Series

A single-valued function $f(z)$ that is analytic at a point $z = z_0$ can be expanded in a neighborhood of this point in the *Taylor series*

$$f(z)=\sum_{n=0}^{\infty}c_n(z-z_0)^n, \tag{7.8}$$

whose expansion coefficients c_n are given by the following formulas:

$$c_n=\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)\,dz}{(z-z_0)^{n+1}}=\frac{f^{(n)}(z_0)}{n!}\quad (n=0,\ 1,\ 2,\ \ldots), \tag{7.9}$$

with Γ a circle centered at $z = z_0$ and lying entirely in the neighborhood of the point z_0 at which the function $f(z)$ is analytic. The center of the circle of convergence is at z_0 and the edge of the circle passes through the singular point ζ of $f(z)$ that is nearest to z_0, i.e. the radius of convergence of series (7.8) is equal to the distance between z_0 and the nearest singular point of $f(z)$.

For the functions $\ln(1+z)$ and $(1+z)^{\alpha}$ the Taylor expansions in the neighborhood of point $z_0 = 0$ are

$$\ln(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\ldots+(-1)^{n-1}\frac{z^n}{n}+\ldots \quad (R=1) \tag{7.10}$$

$$(1+z)^{\alpha}=1+\alpha z+\frac{\alpha(\alpha-1)}{2!}z^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3+\ldots$$
$$\ldots+\frac{\alpha(\alpha-1)\ldots(\alpha+n-1)}{n!}z^n+\ldots\quad (R=1). \tag{7.11}$$

For instance, for $\alpha = -1$ we have

$$\frac{1}{1+z}=1-z+z^2-\ldots+(-1)^nz^n+\ldots\quad (R=1). \tag{7.12}$$

Formula (7.10) gives the expansion into a Taylor series in the neighborhood of $z = 0$ of the principal value of the logarithm; to obtain Taylor series for other values of the multiple-valued function Ln $(1 + z)$ we must add to the series in (7.10) the terms $2n\pi i$, $n = \pm 1, \pm 2, \ldots$:

$$\operatorname{Ln}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots + 2n\pi i.$$

Example 5. Expand into a Taylor series in the neighborhood of point $z_0 = 0$ the function

$$f(z) = \frac{z}{z^2 - 2z - 3},$$

using the expansion (7.12), and find the radius of convergence of this series.

Solution. Let us expand the given function into partial fractions:

$$\frac{z}{z^2-2z-3} = -\frac{1}{4}\,\frac{1}{z+1} + \frac{3}{4}\,\frac{1}{z-3}.$$

We write the right-hand side in the following way:

$$f(z) = \frac{1}{4}\,\frac{1}{1+z} - \frac{1}{4}\,\frac{1}{1-\frac{z}{3}}.$$

Employing the expansion (7.12), we obtain

$$\begin{aligned} f(z) &= \frac{1}{4}(1 - z + z^2 - z^3 + \ldots) - \frac{1}{4}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \ldots\right) \\ &= \frac{1}{4}\left(-\frac{4}{3}z + \frac{8}{9}z^2 - \frac{28}{27}z^3 + \ldots\right) \\ &= -\frac{z}{3} + \frac{2}{3^2}z^2 - \frac{7}{3^3}z^3 + \ldots. \end{aligned}$$

The singular point of the given function that is nearest to $z_0 = 0$ is point $z = -1$. Therefore, the radius of convergence of the series is $R = 1$.

Example 6. Expand in powers of $z - 3$ the function

$$f(z) = \frac{1}{3-2z}.$$

Solution. Let us transform $f(z)$ as follows:

$$\frac{1}{3-2z} = \frac{1}{3-2(z-3+3)} = \frac{1}{-3-2(z-3)} = -\frac{1}{3}\,\frac{1}{1+\frac{2}{3}(z-3)}.$$

Substituting $2(z-3)/3$ for z in (7.12), we obtain

$$\frac{1}{3-2z} = -\frac{1}{3}\left[1-\frac{2}{3}(z-3)+\frac{2^2}{3^2}(z-3)^2-\frac{2^3}{3^3}(z-3)^3+\ldots\right]$$

$$= -\frac{1}{3}-\frac{2}{3^2}(z-3)-\frac{2^2}{3^3}(z-3)^2+\frac{2^3}{3^4}(z-3)^3-\ldots .$$

This series converges when

$$\left|\frac{2}{3}(z-3)\right|<1$$

or $|z-3|<3/2$, i.e. the radius of convergence is $R=3/2$.

Example 7. Find the first few terms in the expansion of $f(z)=\tan z$ in powers of z and establish the radius of convergence of the series.

Solution. Suppose the sought series is

$$f(z)=c_0+c_1z+c_2z^2+c_3z^3+\ldots,$$

where

$$c_n=\frac{f^{(n)}(0)}{n!}\quad (n=0,\ 1,\ 2,\ \ldots),\quad f^{(0)}(0)=f(0)=0.$$

To find the values of the derivatives $f^{(n)}(z)$ at $z=0$ we differentiate the function. We have

$$f'(z)=\frac{1}{\cos^2 z}\quad \text{or}\quad f'(z)=1+f^2(z) \tag{7.13}$$

$$\left.\begin{aligned} f''(z)&=2f(z)f'(z),\\ f'''(z)&=2[f'^2(z)+f(z)f''(z)],\\ f^{(IV)}(z)&=2[3f'(z)f''(z)+f(z)f'''(z)],\\ f^{(V)}(z)&=2[3f''^2(z)+4f'(z)f'''(z)+f(z)f^{(IV)}(z)],\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\} \tag{7.14}$$

Putting $z=0$ in (7.13) and (7.14), we obtain

$$f'(0)=1,\ f''(0)=0,\ f'''(0)=2,\ f^{IV}(0)=0,$$
$$f^{(V)}(0)=16,\ \ldots.$$

Substituting these values of the derivatives into the initial series, we obtain

$$\tan z=z+\frac{2}{3!}z^3+\frac{16}{5!}z^5+\ldots. \tag{7.15}$$

The singular point nearest to $z=0$ is $\zeta=\pi/2$. For this reason the radius of convergence R of the series is $\pi/2$.

Expand the functions given below in Taylor series, using the already established expansions, and find the radii of convergence for these series:

210. $\sin(2z+1)$ in powers of $z+1$.

211. $\cos z$ in powers of $z+\pi/4$.

212. e^z in powers of $(2z-1)$.

213. $\frac{1}{3z+1}$ in powers of $z+2$.

214. $\frac{z+1}{z^2+4z-5}$ in powers of z.

215. $\frac{z}{z^2+i}$ in powers of z.

216. $\cos^2(iz/2)$ in powers of z.

217. $\sinh^2(z/2)$ in powers of z.

218. $\ln(2-z)$ in powers of z.

219. $\ln(2+z-z^2)$ in powers of z.

Find the first few terms in the Taylor expansions in powers of z of the functions given below. Establish the radii of convergence of the series.

220. $\frac{1}{1+e^z}$. 221. $\frac{1}{2+\sin z}$. 222. $\frac{1}{e^{-z}+5}$.

223. $\ln(1+e^{-z})$.

224. $\ln\cos z$. 225. $\ln(1+\cos z)$. 226. $e^{\frac{1}{1-z}}$.

227. Find the function $f(z)$ that is analytic in the circle $|z| \leqslant 1$ and assumes on the circumference $|z| = 1$ the following value:

$$\frac{a-\cos\theta+i\sin\theta}{a^2-2a\cos\theta+1}, \quad a>1, \quad \theta=\arg z.$$

228. Suppose the function $f(z)=\sum_{k=0}^{\infty} a_k z^k$ is analytic in the circle $|z| \leqslant 1$. Prove that the mean value of the function $f(z)/z^n$ on the circumference $|z| = 1$ is a_n.

Suppose we have the series

$$\frac{c_{-1}}{z-z_0}+\frac{c_{-2}}{(z-z_0)^2}+\ldots+\frac{c_{-n}}{(z-z_0)^n}+\ldots=\sum_{n=1}^{\infty}\frac{c_{-n}}{(z-z_0)^n}. \quad (7.16)$$

If $c_{-n} \neq 0$ and there is the finite limit

$$r = \lim_{n\to\infty} \frac{|c_{-n-1}|}{|c_{-n}|}, \tag{7.17}$$

the series converges in the domain

$$|z - z_0| > r. \tag{7.18}$$

Example 8. Find the domain of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(1+i)^{n+1}}{z^n}.$$

Solution. Here $c_{-n} = (1+i)^{n+1}$, $c_{-n-1} = (1+i)^{n+2}$, and $z_0 = 0$. Whence,

$$r = \lim_{n\to\infty} \frac{|(1+i)^{n+2}|}{|(1+i)^{n+1}|} = \lim_{n\to\infty} |1+i| = \sqrt{2}.$$

The domain of convergence is $|z| > \sqrt{2}$.

Example 9. Find the circle of convergence of

$$\sum_{n=1}^{\infty} \frac{\sin in}{(z+i)^n}.$$

Solution. We have

$$c_{-n} = \sin in = i \sinh n, \quad c_{-n-1} = i \sinh (n+1).$$

Whence,

$$r = \lim_{n\to\infty} \frac{|i \sinh (n+1)|}{|i \sinh n|} = \lim_{n\to\infty} \frac{\sinh (n+1)}{\sinh n}$$

$$= \lim_{n\to\infty} \frac{e^{n+1} - e^{-n-1}}{e^n - e^{-n}} = \lim_{n\to\infty} \frac{e - e^{-2n-1}}{1 - e^{-2n}} = e.$$

Hence the domain of convergence of the series is $|z + i| > e$, which means that the series converges in exterior of the circle with radius e centered at point $z_0 = -i$.

Find the circles of convergence of the following series:

229. $\sum_{n=1}^{\infty} \frac{1}{(1-i)^n z^n}$.

230. $\sum_{n=1}^{\infty} \frac{(\sqrt{2} + i\sqrt{2})^n}{z^n}$. **231.** $\sum_{n=1}^{\infty} \frac{z^{-n}}{\cos in}$.

232. $\sum_{n=1}^{\infty} e^n (iz)^{-n}$. 233. $\sum_{n=1}^{\infty} \frac{1}{4^n (z+1)^n}$.

234. $\sum_{n=1}^{\infty} \frac{n2^{-n}}{(z-2-i)^n}$. 235. $\sum_{n=1}^{\infty} \frac{3^n+1}{(z+2i)^n}$.

236. $\sum_{n=1}^{\infty} \frac{(z+1-i)^{-n}}{n+i}$.

The series of the type

$$\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-z_0)^n} + \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

$$= \ldots + \frac{c_{-n}}{(z-z_0)^n} + \ldots + \frac{c_{-1}}{z-z_0} + c_0$$

$$+ c_1 (z-z_0) + \ldots + c_n (z-z_0)^n + \ldots \quad (7.19)$$

converges in the domain where the two series given below converge:

$$\sum_{n=1}^{\infty} \frac{c_{-n}}{(z-z_0)^n} = \frac{c_{-1}}{z-z_0} + \frac{c_{-2}}{(z-z_0)^2} + \ldots, \quad (7.20)$$

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n = c_0 + c_1 (z-z_0) + c_2 (z-z_0)^2 + \ldots. \quad (7.21)$$

Suppose series (7.20) converges in the domain $|z - z_0| > r$, i.e. in the exterior of the circle of radius r centered at point $z = z_0$, and series (7.21) in the circle $|z - z_0| < R$. Then,

(1) if $r > R$, (7.19) is everywhere convergent;

(2) if $r < R$, (7.19) is convergent in the annulus $r < |z - z_0| < R$. Here $r \geqslant 0$ and $0 < R \leqslant +\infty$.

Example 10. Determine the domain of convergence of the series

$$\sum_{n=1}^{\infty} \frac{e^{in}}{(z+1)^n} + \sum_{n=0}^{\infty} \frac{(z+1)^n}{e^{in+1/2}}.$$

Solution. For the series $\sum_{n=1}^{\infty} \frac{e^{in}}{(z+1)^n}$ we have $c_{-n} = e^{in}$ and $c_{-n-1} = e^{i(n+1)}$. Consequently, $r = \lim_{n\to\infty} \frac{|e^{i(n+1)}|}{|e^{in}|} = 1$, so that the first series converges in the domain $|z+1|>1$. For the series $\sum_{n=0}^{\infty} \frac{(z-z_0)^n}{e^{in+1/2}}$ we have $c_n = e^{-in-1/2}$ and $c_{n+1} = e^{-i(n+1)-1/2}$. Its radius of convergence is

$$R = \lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} = \lim_{n\to\infty} \frac{|e^{-in-1/2}|}{|e^{-i(n+1)-1/2}|} = 1,$$

so that the second series converges in the circle $|z+1|<1$. Therefore, the given series diverges everywhere.

Example 11. Find the domain of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(3+4i)^n}{(z+2i)^n} + \sum_{n=0}^{\infty} \left(\frac{z+2i}{6}\right)^n.$$

Solution. For the series $\sum_{n=1}^{\infty} \frac{(3+4i)^n}{(z+2i)^n}$ we have $c_{-n} = (3+4i)^n$ and $c_{-n-1} = (3+4i)^{n+1}$. Consequently,

$$r = \lim_{n\to\infty} \frac{|(3+4i)^{n+1}|}{|(3+4i)^n|} = \lim_{n\to\infty} |3+4i| = 5.$$

Hence, the first series converges in the domain $|z+2i|>5$. For $\sum_{n=0}^{\infty} \left(\frac{z+2i}{6}\right)^n$ we have $c_n = 6^{-n}$ and $c_{n+1} = 6^{-n-1}$. Whence the radius of convergence of this power series is

$$R = \lim_{n\to\infty} \frac{|6^{-n}|}{|6^{-n-1}|} = 6,$$

and it converges in the circle $|z+2i|<6$. Thus, $r = 5 < R = 6$. The given series therefore is convergent in the annulus $5 < |z+2i| < 6$.

Find the domains of convergence of the following series:

237. $\sum_{n=1}^{\infty} \left(\frac{1}{n} + in\right)(z+1+i)^n.$

238. $\sum_{n=1}^{\infty} \frac{(z-i)^n}{2^n}.$

239. $\sum_{n=1}^{\infty} \frac{1}{n^n (z-2+i)^n} + \sum_{n=0}^{\infty} (1+in)(z-2+i)^n.$

240. $\sum_{n=1}^{\infty} \left(\frac{2}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n.$

241. $\sum_{n=1}^{\infty} \frac{n}{(z+1-i)^n} + \sum_{n=0}^{\infty} n(z+1-i)^n.$

242. $\sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}.$

243. $\sum_{n=1}^{\infty} \frac{\sin in}{(z-i)^n} + \sum_{n=0}^{\infty} \frac{(z-i)^n}{n!} \quad (0! = 1).$

244. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 z^n} + \sum_{n=1}^{\infty} \frac{z^n}{n 2^n}.$

245. $\sum_{n=1}^{\infty} \frac{2^n - 1}{(z+1)^n} + \sum_{n=0}^{\infty} \frac{(z+1)^n}{(i+n)^n}.$

246. $-\frac{i}{2(z-i)} + \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(2i)^n}.$

247. $\frac{1}{z} + \sum_{n=0}^{\infty} z^n.$

248. $-\frac{1}{z-1} + \sum_{n=0}^{\infty} (-1)^n (z-1)^n.$

249. $\sum_{n=1}^{\infty} \frac{a^n}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{b^n}$ $(b \neq 0)$.

A function $f(z)$ that is single-valued and analytic in the annulus $r < |z - z_0| < R$ (including the cases where $r = 0$ and $R = +\infty$) can be expanded in this annulus in a Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=-\infty}^{-1} c_n (z-z_0)^n + \sum_{n=0}^{\infty} c_n (z-z_0)^n. \tag{7.22}$$

where the expansion coefficients are

$$c_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)\,dz}{(z-z_0)^{n+1}} \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{7.23}$$

Here Γ is an arbitrary circle centered at z_0 and lying entirely inside the annulus. In (7.22) the series

$$\sum_{n=-\infty}^{-1} c_n (z-z_0)^n = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-z_0)^n}$$

is called the *principal part* of the Laurent series, while the series

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n$$

is called the *regular part* of the Laurent series. For practical purposes formula (7.23) for c_n is usually avoided since it leads to cumbersome calculations. The common practice is to use, when possible, the known expansions into Taylor series of elementary functions.

Example 12. Expand the following function in a Laurent series in the annulus $0 < |z - 1| < 2$:

$$f(z) = \frac{1}{(z^2-1)^2}.$$

Solution. The function $f(z) = (z^2 - 1)^{-2}$ is analytic in the annulus $0 < |z - 1| < 2$. The expansion coefficients

in the appropriate Laurent series can be found via (7.23)

$$c_n = \frac{1}{2\pi i} \int_\Gamma \frac{\frac{1}{(z^2-1)^2}}{(z-1)^{n+1}}\, dz = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{(z-1)^{n+3}(z+1)^2},$$

where Γ is an arbitrary circle centered at $z_0 = 1$ and lying in the given annulus. If $n + 3 \leqslant 0$, i.e. $n \leqslant -3$, the integrand $(z-1)^{-n-3}(z+1)^{-2}$ is analytic at all points in the interior of Γ, including point $z = 1$. In this case

$$\int_\Gamma \frac{dz}{(z-1)^{n+3}(z+1)^2} = 0,$$

i.e. $c_n = 0$ at $n = -3, -4, \ldots$. If $n + 3 > 0$, i.e. $n > -3$, then, using formula (6.2) for a derivative of arbitrary order of an analytic function, we obtain

$$c_n = \frac{1}{2\pi i} \int_\Gamma \frac{\frac{1}{(z+1)^2}}{(z-1)^{n+3}}\, dz = \frac{1}{(n+2)!} \frac{d^{n+2}}{dz^{n+2}} \left[\frac{1}{(z+1)^2}\right]\Big|_{z=1}$$

$$= \frac{1}{(n+2)!} \cdot \frac{(-1)^n (n+3)!}{(z+1)^{n+4}}\Big|_{z=1} = \frac{(-1)^n (n+3)}{2^{n+4}}.$$

For $n = -2, -1, 0, 1, 2, \ldots$ we have

$$c_n = \frac{(-1)^n (n+3)}{2^{n+4}}.$$

The Laurent series for the given function in the annulus $0 < |z-1| < 2$ has the form

$$\frac{1}{(z-1)^2} = \sum_{n=-2}^{+\infty} c_n (z-1)^n = \sum_{n=-2}^{+\infty} \frac{(-1)^n (n+3)}{2^{n+4}} (z-1)^n,$$

or

$$\frac{1}{(z^2-1)^2} = \frac{1}{4}\frac{1}{(z-1)^2} - \frac{1}{4}\frac{1}{z-1} + \frac{3}{16} - \frac{1}{8}(z-1)$$

$$+ \frac{5}{64}(z-1)^2 - \frac{3}{64}(z-1)^3 + \ldots.$$

Alternative solution. We wish to represent $f(z)$ in the form of the sum of powers (both positive and negative) of

the difference $z-1$. To this end we rewrite $f(z)$ as follows:

$$f(z)=\frac{1}{(z^2-1)^2}=\frac{1}{4}\left(\frac{1}{z-1}-\frac{1}{z+1}\right)^2$$
$$=\frac{1}{4}\frac{1}{(z-1)^2}-\frac{1}{4}\frac{1}{z-1}+\frac{1}{4}\frac{1}{z+1}+\frac{1}{4}\frac{1}{(z+1)^2}. \quad (7.24)$$

The first two terms on the right-hand side have the required form since they are powers of $z-1$. The other two terms can be written thus:

$$\frac{1}{z+1}=\frac{1}{(z-1)+2}=\frac{1}{2}\frac{1}{1+\frac{z-1}{2}},$$

$$\frac{1}{(z+1)^2}=\frac{1}{4}\left[1+\left(\frac{z-1}{2}\right)\right]^{-2}.$$

Employing (7.12) and then (7.11) with $\alpha=-2$, we obtain

$$\frac{1}{z+1}=\frac{1}{2}\left[1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{2}\right)^3+\ldots\right], \quad (7.25)$$

$$\frac{1}{(z+1)^2}=\frac{1}{4}\left[1-2\cdot\frac{z-1}{2}+\frac{-2(-2-1)}{2!}\left(\frac{z-1}{2}\right)^2\right.$$
$$\left.+\frac{-2(-2-1)(-2-2)}{3!}\left(\frac{z-1}{2}\right)^3+\ldots\right]. \quad (7.26)$$

Substituting (7.25) and (7.26) into (7.24), we find that

$$\frac{1}{(z^2-1)^2}=\frac{1}{4}\frac{1}{(z-1)^2}-\frac{1}{4}\frac{1}{z-1}$$
$$+\frac{1}{8}\left[1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{2}\right)^3+\ldots\right]$$
$$+\frac{1}{16}\left[1-(z-1)+\frac{3}{2^2}(z-1)^2\right.$$
$$\left.-\frac{4}{2^3}(z-1)^3+\ldots\right],$$

or

$$\frac{1}{(z^2-1)^2}=\frac{1}{4}\frac{1}{(z-1)^2}-\frac{1}{4}\frac{1}{z-1}+\frac{3}{16}-\frac{1}{8}(z-1)$$
$$+\frac{5}{64}(z-1)^2-\frac{3}{64}(z-1)^3+\ldots.$$

Example 13. Expand in a Laurent series the function

$$f(z) = z^2 \cos(1/z)$$

in the neighborhood of point $z_0 = 0$.

Solution. For ζ complex we have

$$\cos\zeta = 1 - \frac{\zeta^2}{2!} + \frac{\zeta^4}{4!} - \frac{\zeta^6}{6!} + \ldots.$$

Assuming that $\zeta = 1/z$, we obtain

$$z^2 \cos\frac{1}{z} = z^2\left(1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \frac{1}{6!z^6} + \ldots\right)$$

$$= z^2 - \frac{1}{2!} + \frac{1}{4!z^2} - \frac{1}{6!z^4} + \ldots,$$

or

$$z^2 \cos\frac{1}{z} = -\frac{1}{2} + z^2 + \frac{1}{4!z^2} - \frac{1}{6!z^4} + \ldots.$$

This expansion is valid for all nonzero z. In the case at hand the annulus is the entire complex plane without one point, $z = 0$. The annulus is given by the following relationship: $0 < |z - 0| < +\infty$. Here $r = 0$, $R = +\infty$, and $z_0 = 0$. The given function is analytic in the noted annulus.

Example 14. Examine the various Laurent expansions of the function

$$f(z) = \frac{2z+1}{z^2+z-2},$$

assuming that $z_0 = 0$.

Solution. The function $f(z)$ has two singular points, $z_1 = -2$ and $z_2 = 1$. Hence, there are three annului centered at $z_0 = 0$ in each of which $f(z)$ is analytic:

(a) the circle $|z| < 1$;

(b) the annulus $1 < |z| < 2$;

(c) $2 < |z| < +\infty$ (the exterior of the circle $|z| \leqslant 2$). Let us find the Laurent series of $f(z)$ in each annulus. We write $f(z)$ as a sum of partial fractions:

$$f(z) = \frac{1}{z+2} + \frac{1}{z-1}. \qquad (7.27)$$

(a) Expansion in the circle $|z| < 1$. We transform (7.27) in the following way:

$$f(z) = \frac{1}{z-1} + \frac{1}{z+2} = \frac{1}{2}\frac{1}{1+\frac{z}{2}} - \frac{1}{1-z}. \qquad (7.28)$$

Employing (7.12), we obtain

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots, \quad |z| < 1, \tag{7.29}$$

$$\frac{1}{1+\frac{z}{2}} = 1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \ldots, \quad |z| < 2 \tag{7.30}$$

Substituting (7.29) and (7.30) into (7.28), we obtain

$$\frac{2z+1}{z^2+z-2} = \frac{1}{2} - \frac{z}{4} + \frac{z^2}{8} - \frac{z^3}{16} + \ldots - (1 + z + z^2 + z^3 + \ldots)$$

$$= -\frac{1}{2} - \frac{3}{4}z - \frac{7}{8}z^2 - \frac{15}{16}z^3 + \ldots$$

which is the Taylor expansion of $f(z)$.

(b) Expansion in the annulus $1 < |z| < 2$. The series (7.30) representing $(1 + z/2)^{-1}$ remains convergent in this annulus, since $|z| < 2$, while the series (7.29) representing $(1 - z)^{-1}$ is divergent for $|z| > 1$. For this reason we rewrite $f(z)$ as follows:

$$f(z) = \frac{1}{2}\frac{1}{1+\frac{z}{2}} + \frac{1}{z}\frac{1}{1-\frac{1}{z}}. \tag{7.31}$$

Applying (7.12), we obtain

$$\frac{1}{1-\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots. \tag{7.32}$$

This series converges at $|1/z| < 1$, i.e. at $|z| > 1$. Substituting (7.30) and (7.32) into (7.31), we find that

$$\frac{2z+1}{z^2+z-2} = \frac{1}{2} - \frac{z}{4} + \frac{z^2}{8} - \frac{z^3}{26} + \ldots + \frac{1}{z} + \frac{1}{z^2} + \ldots$$

$$= \ldots + \frac{1}{z^3} + \frac{1}{z} + \frac{1}{2} - \frac{z}{4} + \frac{z^2}{8} - \frac{z^3}{16} + \ldots,$$

or

$$\frac{2z+1}{z^2+z-2} = \sum_{n=1}^{\infty} \frac{1}{z^n} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{z^n}{2^n}.$$

(c) Expansion for $|z| > 2$. The series (7.30) representing $(1 + z/2)^{-1}$ is divergent for $|z| > 2$, while the series (7.32) representing $(1 - 1/z)^{-1}$ is convergent since if $|z| >$

> 2 then surely $|z| > 1$. We write the function $f(z)$ in the following form:

$$f(z) = \frac{1}{z}\frac{1}{1+\frac{2}{z}} + \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\left(\frac{1}{1+\frac{2}{z}} + \frac{1}{1-\frac{1}{z}}\right).$$

Employing (7.12), we obtain

$$f(z) = \frac{1}{z}\left(1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \ldots + 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right)$$

$$= \frac{1}{z}\left(2 - \frac{1}{z} + \frac{5}{z^2} - \frac{7}{z^3} + \ldots\right),$$

or

$$\frac{2z+1}{z^2+z-2} = \frac{2}{z} - \frac{1}{z^2} + \frac{5}{z^3} - \frac{7}{z^4} + \ldots.$$

This example shows that the Laurent series of a function $f(z)$ may have different forms depending on the specific annulus.

Example 15. Expand in a Laurent series the function

$$f(z) = \frac{2z-3}{z^2-3z+2}$$

in the neighborhoods of the singular points of $f(z)$.

Solution. The singular points in this case are $z_1 = 1$ and $z_2 = 2$.

(1) Expansion of $f(z)$ in the neighborhood of the singular point $z_1 = 1$, i.e. in the annulus $0 < |z - 1| < 1$. We write $f(z)$ as a sum of partial fractions:

$$\frac{2z-3}{z^2-3z+2} = \frac{1}{z-1} + \frac{1}{z-2}.$$

Next we write the right-hand side as

$$\frac{2z-3}{z^2-3z+2} = \frac{1}{z-1} - \frac{1}{1-(z-1)}.$$

Applying the expansion in (7.12), in which $-(z-1)$ is substituted for z, we obtain

$$\frac{2z-3}{z^2-3z+2} = \frac{1}{z-1} - [1 + (z-1) + (z-1)^2 + \ldots],$$

or

$$\frac{2z-3}{z^2-3z+2} = \frac{1}{z-1} - \sum_{n=0}^{\infty} (z-1)^n. \qquad (7.33)$$

(2) Expansion of $f(z)$ in the neighborhood of the singular point $z_2 = 2$, i.e. in the annulus $0 < |z-2| < 1$. We have

$$\frac{2z-3}{z^3-3z+2} = \frac{1}{z-1} + \frac{1}{z-2} = \frac{1}{z-2} + \frac{1}{1+(z-2)}$$
$$= \frac{1}{z-2} + 1 - (z-2) + (z-2)^2 - (z-3)^3 + \ldots,$$

or

$$\frac{2z-3}{z^2-3z+2} = \frac{1}{z-2} + \sum_{n=0}^{\infty} (-1)^n (z-2)^n. \qquad (7.34)$$

Expand in Laurent series at $z = 0$ the following functions:

250. $\frac{\sin z}{z^2}$. 251. $\frac{\sin^2 z}{z}$. 252. $\frac{e^z}{z}$. 253. $\frac{e^z}{z^3}$.

254. $z^3 e^{\frac{1}{z}}$. 255. $z^4 \cos\frac{1}{z}$. 256. $\frac{1}{z}\sin^2\frac{2}{z}$.

257. $\frac{1-\cos z}{z^2}$. 258. $\frac{e^z-1}{z}$. 259. $\frac{1+\cos z}{z^4}$.

260. $\frac{1-e^{-z}}{z^3}$.

Expand in Laurent series the following functions at the points stated:

261. $\frac{z}{(z+1)^2}$, $z_0 = -1$. 262. $\frac{\sin z}{z-2}$, $z_0 = 2$.

263. $ze^{\frac{1}{z+i}}$, $z_0 = -i$.

Expand in Laurent series the following functions in the annuli stated:

264. $\frac{1}{(z-2)(z-3)}$, (a) $2 < |z| < 3$; (b) $3 < |z| < +\infty$.

265. $\frac{1}{z^2+z}$, (a) $0 < |z| < 1$; (b) $1 < |z| < +\infty$.

266. $\frac{1}{(z+2)(1+z^2)}$, (a) $1 < |z| < 4$; (b) $4 < |z| < +\infty$.

267. $\frac{2z+3}{z^2+3z+2}$, $1<|z|<2$.

268. $\frac{z^2-z+3}{z^3-3z+2}$, (a) $|z|<1$; (b) $1<|z|<2$; (c) $2<|z|<\infty$.

269. $\frac{2}{z^2-1}$, $1<|z-2|<3$.

270. $\frac{1}{z^2+2z-8}$, $1<|z+2|<4$.

271. $\frac{z+2}{z^2-4z+3}$, $2<|z-1|<+\infty$.

272. $\frac{z^5}{(z^2-4)^2}$, $2<|z|<+\infty$.

273. $\frac{z}{(z^2-4)(z^2-1)}$, $1<|z|<2$.

274. $\frac{1}{z^2+1}$, $0<|z-i|<2$.

275. $\frac{1}{(z^2-4)^2}$, $4<|z+2|<+\infty$.

§ 8 Zeros of a Function. Isolated Singularities

Zeros of a Function

Suppose $f(z)$ is a function that is analytic at z_0. This point is said to be a *zero of $f(z)$ of the nth order* if

$$f(z_0)=0,\quad f'(z_0)=0,\ \ldots,\ f^{(n-1)}(z_0)=0,\quad f^{(n)}(z_0)\neq 0.$$

If $n=1$, point z_0 is said to be a *simple zero of $f(z)$*.

A point z_0 is a zero of a function $f(z)$ that is analytic at z_0 if and only if in a neighborhood of this point

$$f(z)=(z-z_0)^n\varphi(z),$$

where $\varphi(z)$ is analytic at z_0, and $\varphi(z_0)\neq 0$.

Example 1. Find the zeros of $f(z)=1+\cos z$ and determine their order.

Solution. Nullification of $f(z)$ yields $\cos z=-1$, which implies that the points $z_n=(2n+1)\pi$ ($n=0,\pm 1,\pm 2,\ldots$) are the zeros of $f(z)$. Moreover,

$$f'[(2n+1)\pi]=-\sin(2n+1)\pi=0,$$
$$f''[(2n+1)\pi]=-\cos(2n+1)\pi=1\neq 0.$$

which means that the $z_n = (2n + 1)\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$ are second-order zeros of $f(z)$.

Example 2. Find the zeros of $f(z) = 1 - e^z$ and determine their order.

Solution. Nullification of $f(z)$ yields the zeros $z_n = 2n\pi i$ $(n = 0, \pm 1, \ldots)$ of the function $f(z)$. Moreover,

$$f'(2n\pi i) = -e^{2n\pi i} = -1 \neq 0.$$

Thus, $f(2n\pi i) = 0$ while $f'(2n\pi i) \neq 0$, which implies that the $z_n = 2n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$ are simple zeros of $f(z) = 1 - e^z$.

Example 3. Find the order of the zero $z_0 = 0$ of the function

$$f(z) = \frac{z^8}{z - \sin z}.$$

Solution. Employing the Taylor expansion of $\sin z$ at $z_0 = 0$, we find that

$$f(z) = \frac{z^8}{z - \sin z} = \frac{z^8}{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)}$$

$$= \frac{z^8}{\frac{z^3}{3!} - \frac{z^5}{5!} + \ldots} = \frac{z^5}{\frac{1}{3!} - \frac{z^2}{5!} + \ldots} = z^5 \frac{1}{\frac{1}{3!} - \frac{z^2}{5!} + \ldots}.$$

We put

$$\varphi(z) = \frac{1}{\frac{1}{3!} - \frac{z^2}{5!} + \ldots}.$$

Then $f(z) = z^5\varphi(z)$, where $\varphi(z)$ is analytic at $z_0 = 0$, and $\varphi(0) = 6 \neq 0$. Hence, point $z_0 = 0$ is a fifth-order zero of $f(z)$.

Example 4. Find the zeros of $f(z) = (z^2 + 1)\sinh z$ and determine their order.

Solution. Putting $f(z) = 0$, we obtain $(z^2 + 1)^3 \sinh z = 0$, whence either $z^2 + 1 = 0$ or $\sinh z = 0$. Solving these two equations, we find the zeros of $f(z)$:

$$z = -i,\ z = i,\ z = k\pi i\ (k = 0, \pm 1, \pm 2, \ldots).$$

We put $z = -i$. We then can write

$$f(z) = (z + i)^3 \varphi(z),$$

where the function $\varphi(z) = (z - i)^3 \sinh z$ is analytic at $z = -i$ and $\varphi(-i) = 8i \sinh i = -8 \sin 1 \neq 0$. This means that the point $z = -i$ is a third-order zero of $f(z)$. Similarly we can prove that point $z = i$ is a third-order zero of $f(z)$, too. Now let us investigate the zeros $z = k\pi i$ ($k = 0, \pm 1, \pm 2, \ldots$). The derivative

$$f'(z) = 6z(z^2 + 1)^2 \sinh z + (z^2 + 1)^3 \cosh z$$

is nonzero at $z = k\pi i$, and hence these points are first-order zeros.

Find the zeros of the following functions and determine their order:

276. (a) $f(z) = z^4 + 4z^2$; (b) $f(z) = (\sin z)/z$.

277. (a) $f(z) = z^2 \sin z$; (b) $f(z) = (\sinh^2 z)/z$.

278. (a) $f(z) = 1 + \cosh z$; (b) $f(z) = (1 - \sinh z)^2/z$.

279. (a) $f(z) = (z + \pi i) \sinh z$; (b) $f(z) = \cos z^3$.

280. (a) $f(z) = (z^2 + \pi^2)(1 + e^{-z})$; (b) $f(z) = \cos z + \cosh iz$.

Find the order of the zero $z_0 = 0$ of the following functions:

281. $f(z) = z^6 [(z/2)^2 - (\sin z/2)^2]^{-1}$.

282. $f(z) = e^{\sin z} - e^{\tan z}$.

283. $f(z) = z^3/(1 + z - e^z)$.

284. $f(z) = 2(\cosh z - 1) - z^2$.

285. $f(z) = (1 - \cos 2z)^2/(z - \sinh z)$.

286. $f(z) = (e^z - e^{z^2}) \ln (1 - z)$.

287. $f(z) = z^2 (e^{z^2} - 1)$.

288. $f(z) = 6 \sin z^3 + z^3 (z^6 - 6)$.

289. Suppose that a point z_0 is an nth order zero for a function $\varphi(z)$ and an mth order zero for a function $\psi(z)$. What is point z_0 for the following functions:

(a) $\varphi(z) + \psi(z)$; (b) $\varphi(z)\psi(z)$; (c) $\varphi(z)/\psi(z)$?

Isolated Singular Points

A point z_0 is said to be an *isolated singular point*, or simply an *isolated singularity*, of a function if there is a neighborhood of this point in which the function is analytic everywhere except at $z = z_0$. An isolated singular point of a function $f(z)$ is said to be *removable* if at this point $f(z)$ has a finite limit.

Example 5. $f(z) = (e^z - 1)/z$. The singular point of $f(z)$ is $z_0 = 0$. We have

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{e^z - 1}{z} = 1.$$

Hence, $z_0 = 0$ is a removable singular point.

Point z_0 is said to be a *pole* of a function $f(z)$ if $f(z)$ tends to ∞ as $z \to z_0$.

A point z_0 is a pole of a function $f(z)$ if and only if it is a zero of the function $\varphi(z) = 1/f(z)$.

A point z_0 is said to be a *pole of the nth order* ($n \geqslant 1$) *of* $f(z)$ if it is a zero of the nth order of $\varphi(z) = 1/f(z)$. When $n = 1$, the pole is said to be *simple.*

Example 6. $f(z) = 1/z^3$. There is one singular point, $z_0 = 0$. We put $z = \rho e^{i\varphi}$. Then $f(z) = e^{-i3\varphi}/\rho^3$. Obviously, $|f(z)| = 1/\rho^3$, which means that $|f(z)|$ increases without limit as $z \to 0$ in any manner. Hence, $f(z)$ tends to ∞ as $z \to 0$, i.e. point $z_0 = 0$ is the pole of this function. This point is a third-order zero for the function $\varphi(z) = z^3$, which means it is a third-order pole for the function $f(z) = 1/z^3$.

A point z_0 is an nth order pole of a function $f(z)$ if and only if we can write $f(z) = \varphi(z)/(z - z_0)^n$, with $\varphi(z)$ analytic at z_0, and $\varphi(z_0) \neq 0$.

Example 7. $f(z) = (\sin z)/(z^3 + z^2 - z - 1)$. This function has two singular points, $z = -1$ and $z = 1$. We take point $z = -1$. We write

$$f(z) = \frac{\dfrac{\sin z}{z-1}}{(z+1)^2}.$$

Here

$$\varphi(z) = \frac{\sin z}{z-1}$$

is analytic in the neighborhood of point $z = -1$, and $\varphi(-1) = (\sin 1)/2 \neq 0$. Hence, point $z = -1$ is a second-order pole of the given function. Similarly, if we write $f(z)$ in the form

$$f(z) = \frac{\dfrac{\sin z}{(z+1)^2}}{z-1},$$

we can conclude that point $z = 1$ is a simple pole of $f(z)$.

A point z_0 is said to be an *essential singularity of a function* $f(z)$ if there is no limit for $f(z)$ (either finite or infinite) as $z \to z_0$.

Example 8. Determine the type of the singular point $z = 0$ of the function $f(z) = e^{1/z^2}$.

Solution. Let us see how this function behaves on the real and imaginary axes. On the real axis $z = x$ and $f(x) = e^{1/x^2}$, which tends to ∞ as $x \to 0$. On the imaginary axis $z = iy$ and $f(iy) = e^{-1/y^2}$, which tends to zero as $y \to 0$. Therefore, $f(z)$ has no limit, either finite or infinite, at point $z = 0$. This means that point $z = 0$ is an essential singularity for $f(z)$.

Example 9. Determine the type of the singular point $z = 0$ of the function

$$f(z) = \frac{1}{2 + z^2 - 2\cosh z}.$$

Solution. Point $z = 0$ is a pole for $f(z)$, since it is a zero of the denominator of $f(z)$. We take the function $\varphi(z) = 1/f(z) = 2 + z^2 - 2\cosh z$. For it $\varphi(0) = 0$. Let us find the order of the zero $z = 0$ of this function. Precisely,

$$\begin{aligned}
\varphi'(z) &= 2z - 2\sinh z, & \varphi'(0) &= 0;\\
\varphi''(z) &= 2 - 2\cosh z, & \varphi''(0) &= 0;\\
\varphi'''(z) &= -2\sinh z, & \varphi'''(0) &= 0;\\
\varphi^{(IV)}(z) &= -2\cosh z, & \varphi^{(IV)}(0) &= -2 \neq 0.
\end{aligned}$$

Therefore, point $z = 0$ is a fourth-order zero for $\varphi(z)$, which means that for the given function $f(z)$ point $z = 0$ is a fourth-order pole.

Example 10. Determine the type of the singular point $z = 1$ of the function

$$f(z) = \frac{\sin \pi z}{2e^{z-1} - z^2 - 1}.$$

Solution. We study the function

$$\varphi(z) = \frac{1}{f(z)} = \frac{2e^{z-1} - z^2 - 1}{\sin \pi z}.$$

Point $z = 1$ is a third-order zero for the numerator

$$\psi(z) = 2e^{z-1} - z^2 - 1,$$

since

$$\psi(1)=0;\quad \psi'(1)=(2e^{z-1}-2z)\,|_{z=1}=0;$$
$$\psi''(1)=(2e^{z-1}-2)\,|_{z=1}=0;\quad \varphi'''(1)=2e^{z-1}=2\neq 0.$$

Point $z=1$ is a first-order zero for the denominator $\sin \pi z$ of $\varphi(z)$. Hence, it is a $(3-1=2)$ nd order zero of $\varphi(z)$, which means it is a second-order pole for $f(z)$.

Determine the type of the singular point $z_0=0$ of the following function:

290. (a) $\dfrac{1}{z-\sin z}$; (b) $\dfrac{1}{\cos z-1+\dfrac{z^2}{2}}$; (c) $\dfrac{1}{e^{-z}+z-1}$.

291. (a) $\dfrac{\sin z}{e^{-z}+z-1}$; (b) $\dfrac{\sinh z}{z-\sinh z}$.

Find the singular points and determine their types for the following functions:

292. (a) $\dfrac{1}{1-\sin z}$; (b) $\dfrac{1-\cos z}{z^2}$.

293. (a) $e^{\frac{1}{z+2}}$; (b) $\cos\dfrac{1}{z}$.

294. (a) $\dfrac{z}{z^5+2z^4+z^3}$; (b) $\dfrac{1}{e^{-z}-1}+\dfrac{1}{z^2}$.

295. (a) e^{-1/z^2}; (b) $\sin\dfrac{\pi}{z+1}$; (c) $\cosh(1/z)$.

296. (a) $\dfrac{z}{\cos z-1}$; (b) $\dfrac{1-\sin z}{\cos z}$; (c) $\dfrac{z-\pi}{\sin^2 z}$.

Here are some statements concerning singular points.

(1) *A singular point z_0 of $f(z)$ is removable if and only if the Laurent expansion of $f(z)$ at z_0 has no principal part.*

(2) *A singular point z_0 of $f(z)$ is a pole if and only if the principal part of the Laurent expansion of $f(z)$ at z_0 has a finite number of terms, namely,*

$$f(z)=\frac{c_{-k}}{(z-z_0)^k}+\ldots+\frac{c_{-1}}{z-z_0}+\sum_{n=0}^{\infty}c_n(z-z_0)^n\quad (c_{-k}\neq 0).$$

The greatest power of the difference $z-z_0$ in the denominators of the terms of the principal part of the Laurent series is the order of the pole.

(3) *A singular point z_0 of $f(z)$ is an essential singularity if and only if the principal part of the Laurent expansion of $f(z)$ at z_0 has an infinite number of terms.*

Example 11. Establish the type of the singular point $z_0 = 0$ of the function

$$f(z) = \frac{1 - e^{-z}}{z}.$$

Solution. Employing the Taylor expansion of e^{-z} in the neighborhood of point $z_0 = 0$, we arrive at the Laurent expansion for $f(z)$ in the neighborhood of this point:

$$f(z) = \frac{1}{z}(1 - e^{-z}) = \frac{1}{z}\left[1 - \left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \ldots\right)\right]$$

$$= \frac{1}{z}\left(z - \frac{z^2}{2!} + \frac{z^3}{3!} - \ldots\right) = 1 - \frac{z}{2!} + \frac{z^2}{3!} - \ldots.$$

This expansion has no principal part. Whence point $z_0 = 0$ is a removable singular point. If we put $f(z)$ equal to unity at $z = 0$, i.e.

$$f(z) = \begin{cases} \dfrac{1 - e^{-z}}{z} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0, \end{cases}$$

$f(z)$ becomes analytic at point $z_0 = 0$, too.

Example 12. Determine the type of the singular point $z_0 = 0$ of the function

$$f(z) = \frac{1 - \cos z}{z^7}.$$

Solution. Writing the Taylor expansion for $\cos z$ in powers of z, we arrive at the Laurent expansion for $f(z)$ in the neighborhood of point $z_0 = 0$:

$$f(z) = \frac{1}{z^7}\left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \frac{z^8}{8!} + \frac{z^{10}}{10!} - \ldots\right)$$

$$= \frac{1}{2!z^5} - \frac{1}{4!z^3} + \frac{1}{6!z} - \frac{z}{8!} + \frac{z^3}{10!} - \ldots.$$

The Laurent expansion for the function $f(z)$ in the neighborhood of point $z_0 = 0$ has a finite number of terms with negative powers of z. Therefore, this point is a fifth-order pole, since the greatest power of z in the denominators of the terms of the principal part of the Laurent series for $f(z)$ is five.

Example 13. Determine the type of the singular point $z_0 = 1$ of the function

$$f(z) = (z-1)\, e^{1/(z-1)}.$$

Solution. Using the expansion

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \ldots$$

and assuming that $u = (z-1)^{-1}$, we arrive at the Laurent expansion for $f(z)$ in the neighborhood of point $z_0 = 1$:

$$f(z) = (z-1)\left[1 + \frac{1}{z-1} + \frac{1}{2!\,(z-1)^2} + \frac{1}{3!\,(z-1)^3} + \ldots\right]$$

$$= 1 + (z-1) + \frac{1}{2!\,(z-1)} + \frac{1}{3!\,(z-1)^2} + \ldots.$$

This expansion contains an infinite number of terms with negative powers of $z-1$. Hence, point $z_0 = 1$ is an essential singularity for $f(z)$.

Determine the type of the following singular points:

297. $\dfrac{1+\cos z}{z-\pi}$, $z_0 = \pi$.

298. $\dfrac{z^2-3z+2}{z^2-2z+1}$, $z_0 = 1$.

299. $(\sin z)/z^2$, $z_0 = 0$.

300. $(\sinh z)/z$, $z_0 = 0$.

301. $\cos\dfrac{1}{z+\pi}$, $z_0 = -\pi$.

302. $\dfrac{z^2-1}{z^6+2z^5+z^4}$, $z = 0$, $z_0 = -1$.

303. $\dfrac{\ln(1+z^3)}{z^2}$, $z_0 = 0$. **304.** $\dfrac{\sin^2 z}{z}$, $z_0 = 0$.

305. $\dfrac{e^{z+e}}{z+e}$, $z_0 = -e$.

306. $\cos\dfrac{1}{z} + \sin\dfrac{2-\pi z}{2z}$, $z_0 = 0$.

307. $z \sinh(1/z)$, $z_0 = 0$.

§ 9 Residues of a Function

Suppose a point z_0 is an isolated singular point of a function $f(z)$. The *residue* of $f(z)$ at point z_0 is the quantity

$$\operatorname{Res} f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} f(z)\, dz. \tag{9.1}$$

(Other notations are $\operatorname{Res}_{z=z_0} f(z)$, $\operatorname{Res}[f]_{z_0}$, $R(z_0; f)$, or $R(a)$ if we need not indicate f.) The contour γ is a circle centered at z_0 and so small that it does not exceed the region of analyticity of $f(z)$ and has no other singular points in its interior. The value of the residue is the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion of $f(z)$ about z_0:

$$\operatorname{Res} f(z_0) = c_{-1}. \tag{9.2}$$

The value of the residue at a removable singular point is zero.

If point z_0 is a pole of the nth order of $f(z)$, then

$$\operatorname{Res} f(z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \{f(z)(z - z_0)^n\}. \tag{9.3}$$

When z_0 is a simple pole ($n = 1$),

$$\operatorname{Res} f(z_0) = \lim_{z \to z_0} [f(z)(z - z_0)]. \tag{9.4}$$

If in the neighborhood of point z_0 the function $f(z)$ can be written in the form of the quotient of two analytic functions,

$$f(z) = \frac{\varphi(z)}{\psi(z)},$$

with $\varphi(z_0) \neq 0$, $\psi(z_0) = 0$, and $\psi'(z_0) \neq 0$, i.e. if z_0 is a simple pole of $f(z)$, then

$$\operatorname{Res} f(z_0) = \frac{\varphi(z_0)}{\psi'(z_0)}. \tag{9.5}$$

If point z_0 is an essential singularity of $f(z)$, then to find $\operatorname{Res} f(z_0)$ we must find the coefficient c_{-1} in the Laurent expansion of the function $f(z)$ about z_0; this coefficient gives the value of $\operatorname{Res} f(z_0)$.

Example 1. Find the residues of the function

$$f(z) = \frac{\sin z^2}{z^3 - \frac{\pi}{4} z^2}$$

at its singular points.

Solution. The singular points of $(\sin z^2)/(z^3 - \pi z^2/4)$ are $z = 0$ and $z = \pi/4$. At $z = 0$,

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin z^2}{z^2} \lim_{z\to 0} \frac{1}{z - \frac{\pi}{4}} = -\frac{4}{\pi}.$$

Therefore, point $z = 0$ is a removable singularity of $f(z)$. Whence,

$$\operatorname{Res} f(0) = 0.$$

At point $z = \pi/4$ we have $\lim_{z\to\pi/4} f(z) = \infty$, i.e. point $z = \pi/4$ is a (first-order) pole of $f(z)$. According to (9.4) we have

$$\operatorname{Res} f\left(\frac{\pi}{4}\right) = \lim_{z\to\pi/4} f(z)\left(z - \frac{\pi}{4}\right) = \lim_{z\to\pi/4} \frac{\sin z^2}{z^3 - \frac{\pi}{4} z^2}\left(z - \frac{\pi}{4}\right)$$

$$= \lim_{z\to\pi/4} \frac{\sin z^2}{z^2} = \frac{16}{\pi^2} \sin \frac{\pi^2}{16}.$$

Example 2. Find the residues of the function

$$f(z) = \frac{e^z}{(z+1)^3 (z-2)}$$

at its singular points.

Solution. The singular points of $f(z)$ are $z = -1$ and $z = 2$. Point $z = -1$ is a third-order pole for $f(z)$. According to (9.3) we have

$$\operatorname{Res} f(-1) = \frac{1}{2!} \lim_{z\to -1} \frac{d^2}{dz^2}\left(\frac{e^z}{z-2}\right)$$

$$= \frac{1}{2} \lim_{z\to -1} \frac{(z^2 - 6z + 10)\, e^z}{(z-2)^3} = -\frac{17}{54e}.$$

Point $z = 2$ is a first-order pole, whence by (9.4),

$$\operatorname{Res} f(2) = \lim_{z\to 2} \frac{e^z}{(z+1)^3} = \frac{e^3}{27}.$$

Example 3. Find the residues of the function $f(z) = (z^4 + 1)^{-1}$ at its singular points.

Solution. The singular points of $f(z)$ coincide with the zeros of the denominator, i.e. they are the roots of $z^4 + 1 = 0$. We have

$$z_1 = e^{i\frac{\pi}{4}}, \quad z_2 = e^{i\frac{3\pi}{4}}, \quad z_3 = e^{-i\frac{3\pi}{4}}, \quad z_4 = e^{-i\frac{\pi}{4}}.$$

Employing (9.5), we obtain

$$\operatorname{Res} f(z_1) = \frac{1}{4z^3}\Big|_{z=e^{i\frac{\pi}{4}}} = \frac{1}{4}e^{-i\frac{3\pi}{4}},$$

$$\operatorname{Res} f(z_2) = \frac{1}{4z^3}\Big|_{z=e^{i\frac{3\pi}{4}}} = \frac{1}{4}e^{-i\frac{9\pi}{4}},$$

$$\operatorname{Res} f(z_3) = \frac{1}{4z^3}\Big|_{z=e^{-i\frac{3\pi}{4}}} = \frac{1}{4}e^{i\frac{9\pi}{4}},$$

$$\operatorname{Res} f(z_4) = \frac{1}{4z^3}\Big|_{z=e^{-i\frac{\pi}{4}}} = \frac{1}{4}e^{i\frac{3\pi}{4}}.$$

Example 4. Find the residue of the function $f(z) = z^3 \sin(1/z^2)$ at the singular point of $f(z)$.

Solution. Point $z = 0$ is the singular point of $f(z)$. It is an essential singularity for $f(z)$. Indeed, the Laurent expansion of $f(z)$ about $z = 0$ is

$$f(z) = z^3\left(\frac{1}{z^2} - \frac{1}{3!\,z^6} + \frac{1}{5!\,z^{10}} - \ldots\right)$$
$$= z - \frac{1}{3!\,z^3} + \frac{1}{5!\,z^7} - \ldots,$$

i.e. has an infinite number of terms in its principal part. The residue of $f(z)$ at $z = 0$ is zero since the coefficient c_{-1} in the Laurent expansion of $f(z)$ about this point is zero.

If a function $f(z)$ has the form $f(z) = \varphi(z)/\psi(z)$, where the analytic functions $\varphi(z)$ and $\psi(z)$ have zeros of orders higher than the first at z_0, it is convenient to substitute for $\varphi(z)$ and $\psi(z)$ their Taylor expansions about point z_0.

Example 5. Find the value of the residue at $z = 0$ for the function

$$f(z) = \frac{\sin 3z - 3\sin z}{(\sin z - z)\sin z}.$$

Solution. Point $z = 0$ is a zero for both the numerator $\varphi(z) = \sin 3z - 3\sin z$ and the denominator $\psi(z) = (\sin z - z)\sin z$. Let us find the orders of the zeros of

these functions using the Taylor expansion for $\sin z$ in the neighborhood of point $z = 0$:

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots.$$

We have

$$\varphi(z) = \sin 3z - 3\sin z$$

$$= 3z - \frac{3^3 z^3}{3!} + \frac{3^5 z^5}{5!} - \ldots - 3\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)$$

$$= -\frac{3^3-3}{3!} z^3 + \frac{3^5-3}{5!} z^5 - \ldots = z^3 \varphi_1(z),$$

where

$$\varphi_1(z) = -\frac{3^3-3}{3!} + \frac{3^5-3}{5!} z^2 - \ldots, \quad \varphi_1(0) = -4 \neq 0;$$

$$\psi(z) = (\sin z - z)\sin z = \left(-\frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)\left(z - \frac{z^3}{3!} + \ldots\right)$$

$$= z^4\left(-\frac{1}{3!} + \frac{z^2}{5!} - \ldots\right)\left(1 - \frac{z^2}{3!} + \ldots\right) = z^4 \psi_1(z),$$

where

$$\psi_1(z) = \left(-\frac{1}{3!} + \frac{z^2}{5!} - \ldots\right)\left(1 - \frac{z^2}{3!} + \ldots\right),$$

$$\psi_1(0) = -\frac{1}{6} \neq 0.$$

Consequently,

$$f(z) = \frac{\sin 3z - 3\sin z}{(\sin z - z)\sin z} = \frac{z^3 \varphi_1(z)}{z^4 \psi_1(z)} = \frac{\varphi_1(z)}{z\psi_1(z)},$$

and since $\varphi_1(0) \neq 0$ and $\psi_1(0) \neq 0$, we see that point $z = 0$ is a simple pole for the given function, which means that the residue at this point can be found via formula (9.5):

$$\operatorname*{Res}_{z=0} \frac{\sin 3z - 3\sin z}{(\sin z - z)\sin z} = \lim_{z \to 0} \frac{\varphi_1(z)}{z\psi_1(z)} z = \frac{\varphi_1(0)}{\psi_1(0)} = \frac{-4}{-\frac{1}{6}} = 24.$$

Evaluate the following residues:

308. $\operatorname*{Res}_{z=0} \dfrac{z^{n-1}}{\sin^n z}$ $(n = 1, 2, \ldots)$.

309. $\operatorname*{Res}_{z=0} \dfrac{\sin 2z - 2z}{(1-\cos z)^2}$.

310. $\operatorname{Res}\limits_{z=0} \dfrac{e^z-1-z}{(1-\cos 2z)\sin z}$.

311. $\operatorname{Res}\limits_{z=0} \dfrac{(1-\cosh z)\sinh z}{(1-\cos z)\sin^2 z}$.

312. $\operatorname{Res}\limits_{z=0} \dfrac{z^{n-2}}{\sinh^n z}$ $(n=2, 3, \ldots)$.

313. $\operatorname{Res}\limits_{z=0} \dfrac{z^2}{\cosh z-1-z^2/2}$.

Example 6. Find the residues of $f(z) = e^{1/z}/(1-z)$ at its singular points.

Solution. The singular points of this function are $z=1$ and $z=0$. Since point $z=1$ is a simple pole, we have

$$\operatorname{Res}_{z=1} f(z) = \frac{e^{1/z}}{-1}\Big|_{z=1} = -e.$$

To establish the type of the singular point $z=0$ we expand $f(z)$ in a Laurent series about this point. This yields

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \ldots,$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots.$$

Multiplying these series, we obtain

$$\frac{e^{1/z}}{1-z} = \left(1+\frac{1}{z}+\frac{1}{2!\,z^2}+\frac{1}{3!\,z^3}+\ldots\right)(1+z+z^2+z^3+\ldots)$$

$$= \left(1+\frac{1}{2!}+\frac{1}{3!}+\ldots\right)\frac{1}{z} + c_{-2}\frac{1}{z^2} + \ldots + \text{regular part},$$

here $c_{-k} \neq 0$, $k = 2, 3, \ldots$. Since the principal part of the Laurent expansion contains an infinite number of terms with negative powers of z, point $z=0$ is an essential singularity for the given function. The residue at this point is

$$\operatorname{Res}_{z=0} f(z) = c_{-1} = 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots = e - 1.$$

Example 7. Find the value of the residue of $f(z) = \cos z \sin(1/z)$ at the singular point $z=0$.

Solution. To establish the type of the singular point we expand the given function in a Laurent series about point

$z = 0$. We have

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots, \quad \sin \frac{1}{z} = \frac{1}{z} - \frac{1}{3!\, z^3} + \frac{1}{5!\, z^5} - \ldots$$

Multiplying these series we find that

$$\begin{aligned} f(z) &= \cos z \sin \frac{1}{z} \\ &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right)\left(\frac{1}{z} - \frac{1}{3!\, z^3} + \frac{1}{5!\, z^5} - \ldots\right) \\ &= \left(1 + \frac{1}{2!\, 3!} + \frac{1}{4!\, 5!} + \ldots\right)\frac{1}{z} \\ &\quad - \left(\frac{1}{0!\, 3!} + \frac{1}{2!\, 5!} + \frac{1}{4!\, 6!} + \ldots\right)\frac{1}{z^3} + \ldots + \text{regular part.} \end{aligned}$$

Thus, in the neighborhood of $z = 0$ the Laurent series has the form

$$\cos z \sin \frac{1}{z} = \sum_{n=0}^{\infty} \frac{1}{(2n)!\ (2n+1)!}\frac{1}{z} + c_{-3}\frac{1}{z^3} + \ldots + \text{regular part},$$

where $c_{-(2k-1)} \neq 0$, $k = 1, 2, \ldots$. The principal part of the Laurent expansion, therefore, contains an infinite number of terms, which means that point $z = 0$ is an essential singularity for the given function. The residue at this point is

$$\operatorname*{Res}_{z=0}\left(\cos z \sin \frac{1}{z}\right) = c_{-1} = \sum_{n=0}^{\infty} \frac{1}{(2n)!\ (2n+1)!}$$

Example 8. Find the residue of the function

$$w = z^2 \sin \frac{1}{z+1}$$

at the singular point of w.

Solution. Point $z = -1$ is the singular point of the given function. To establish the type of the singular point we expand w in a Laurent series about $z = -1$. To this end we express z^2 in terms of powers of the difference $z - (-1) = z + 1$. We have

$$z^2 = [(z+1) - 1]^2 = (z+1)^2 - 2(z+1) + 1. \quad (9.6)$$

The Laurent series for $\sin \dfrac{1}{z+1}$ about point $z = -1$ has the form

$$\sin\frac{1}{z+1} = \frac{1}{z+1} - \frac{1}{3!\,(z+1)^3} + \frac{1}{5!\,(z+1)^5} - \ldots. \quad (9.7)$$

Multiplying (9.6) by (9.7), we obtain

$$w = z^2 \sin \frac{1}{z+1}$$

$$= [(z+1)^2 - 2(z+1) + 1] \left[\frac{1}{z+1} - \frac{1}{3!\,(z+1)^3} + \frac{1}{5!\,(z+1)^5} - \ldots \right]$$

$$= \left(1 - \frac{1}{3!}\right) \frac{1}{z+1} + \frac{2}{3!} \frac{1}{(z+1)^2}$$

$$+ \left(\frac{1}{5!} - \frac{1}{3!}\right) \frac{1}{(z+1)^3} + \ldots + [-2 + (z+1)].$$

The Laurent series contains an infinite number of terms with negative powers of $z + 1$, which means that point $z = -1$ is an essential singularity for the given function. The value of the residue at this point is

$$\operatorname*{Res}_{z=-1} w = 1 - \frac{1}{3!} = \frac{5}{6}.$$

Example 9. Find the residue of the function

$$f(z) = e^{1/z^2} \cos z$$

at point $z = 0$.

Solution. Since the residue at point $z = 0$ is the coefficient of z^{-1}, we immediately find that the residue in our case is zero, since $f(z)$ is an even function and, therefore, its Laurent expansion about point $z = 0$ cannot have odd powers of z.

Find the residues at the singular points of the following functions:

314. $f(z) = \dfrac{\tan z}{z^2 - \pi z/4}$. **315.** $f(z) = z^3 e^{1/z}$.

316. $f(z) = \dfrac{\cosh z}{(z^2+1)(z-3)}$.

317. $f(z) = \dfrac{e^z}{1/4 - \sin^2 z}$.

318. $f(z) = \dfrac{e^z}{z^3(z-1)}$. **319.** $f(z) = \dfrac{z}{(z+1)^3(z-2)^2}$.

320. $f(z) = \dfrac{e^{-\frac{1}{z^2}}}{1+z^4}$. **321.** $f(z) = z^2 \sin \dfrac{1}{z}$.

322. $f(z) = \cos \dfrac{1}{z} + z^3$.

323. $f(z)=\dfrac{\sin 2z}{(z+i)\left(z-\frac{i}{2}\right)^2}$.

324. $f(z)=\dfrac{1-\cos z}{z^3(z-3)}$. 325. $f(z)=e^{z^2+\frac{1}{z^2}}$.

326. $f(z)=\dfrac{e^{iz}}{(z^2-1)(z-3)}$.

327. $f(z)=\dfrac{\cos z}{z^3-\frac{\pi}{2}z^2}$. 328. $f(z)=\dfrac{e^{\pi z}}{z-i}$.

329. $f(z)=\dfrac{z^{2n}}{(z-1)^n}$ (n a positive integer).

330. $f(z)=\cot^2 z$. 331. $f(z)=\sin z\cos(1/z)$.

332. $f(z)=e^{\frac{z}{z-1}}$. 333. $f(z)=\dfrac{\sin\frac{1}{z}}{1-z}$.

334. $f(z)=\dfrac{e^{\frac{1}{z}}}{1+z}$. 335. $f(z)=e^{\frac{z^2+1}{z}}$.

336. $f(z)=e^z\sin\dfrac{1}{z}$.

§ 10 The Residue Theorem. Applying Residues to Evaluate Definite Integrals. Summation of Series by Means of Residues

The residue theorem

Theorem *If a function $f(z)$ is analytic on the boundary C of a domain D and everywhere in D except, perhaps, at a finite number of points $z_1, z_2, \ldots, z_n$, then*

$$\int_C f(z)\,dz=2\pi i\sum_{k=1}^{n}\operatorname{Res} f(z_k).$$

Example 1. Evaluate the integral $\displaystyle\int_{|z|=4}\frac{e^z-1}{z^2+z}\,dz$.

Solution. In the circle $|z|<4$ the function $f(z)=(e^z-1)/(z^2+z)$ is analytic everywhere except at points

$z - 0$ and $z = -1$. According to the residue theorem,

$$\int\limits_{|z|=4} \frac{e^z-1}{z^2+z}\,dz = 2\pi i\,(\operatorname{Res} f(0) + \operatorname{Res} f(-1)).$$

Point $z = 0$ is a removable singular point for $f(z)$, since

$$\lim_{z\to 0} \frac{e^z-1}{z(z+1)} = 1.$$

Therefore $\operatorname{Res} f(0) = 0$. Point $z = -1$ is a first-order pole, which means that

$$\operatorname{Res} f(-1) = \lim_{z\to -1} \left\{\frac{e^z-1}{z(z+1)}(z+1)\right\} = 1 - e^{-1}.$$

Hence,

$$\int\limits_{|z|=4} \frac{e^z-1}{z^2+z}\,dz = 2\pi i\,(1-e^{-1}).$$

Example 2. Evaluate the integral $\int\limits_{|z|=2} \tan z\,dz$.

Solution. In the domain D: $|z| < 2$ the function $f(z) = \tan z$ is analytic everywhere except at points $z = \pi/2$ and $z = -\pi/2$, which are simple poles. All the other singular points $z_k = \pi/2 + k\pi$ of $f(z) = \tan z$ lie outside D and for this reason may be not taken into account. We have

$$\operatorname{Res} f\left(\frac{\pi}{2}\right) = \frac{\sin z}{(\cos z)'}\Big|_{z=\pi/2} = -1,$$

$$\operatorname{Res} f\left(-\frac{\pi}{2}\right) = \frac{\sin z}{(\cos z)'}\Big|_{z=-\pi/2} = -1.$$

Whence,

$$\int\limits_{|z|=2} \tan z\,dz = -4\pi i.$$

Example 3. Evaluate the integral $\int\limits_{|z-i|=3/2} \frac{e^{1/z^2}}{z^2+1}\,dz$.

Solution. In the domain D: $|z - i| < 3/2$ the function $f(z) = e^{1/z^2}/(z^2+1)$ has two singular points: $z = i$ is a first-order pole, and $z = 0$ is an essential singularity. By formula (9.5) we have

$$\operatorname{Res} f(i) = \frac{e^{1/z^2}}{2z}\Big|_{z=i} = \frac{e^{-1}}{2i}.$$

To find the value of the residue at point $z = 0$ we must have the Laurent expansion of $f(z)$ about point $z = 0$. But in our case there is no need to find the Laurent expansion, since $f(z)$ is an even function and therefore we can immediately say that its Laurent expansion has only even powers of z and $1/z$. Whence $c_{-1} = 0$ and

$$\operatorname{Res} f(0) = 0.$$

By the residue theorem we have

$$\int\limits_{|z-i|=3/2} \frac{e^{1/z^2}}{z^2+1}\,dz = \frac{\pi}{e}.$$

Example 4. Evaluate the integral $\int\limits_{|z|=2} \frac{1}{z-1} \sin\frac{1}{z}\,dz$.

Solution. In the circle $|z| \leqslant 2$ the integrand has two singular points, $z = 1$ and $z = 0$. It is easy to see that $z = 1$ is a simple pole and whence

$$\operatorname*{Res}_{z=1}\left(\frac{1}{z-1}\sin\frac{1}{z}\right) = \left.\frac{\sin\frac{1}{z}}{(z-1)'}\right|_{z=1} = \sin 1.$$

To establish the type of the singular point $z = 0$ we write the Laurent series for the function $(z-1)^{-1}\sin(1/z)$ in the neighborhood of this point. We have

$$\frac{1}{z-1}\sin\frac{1}{z} = -\frac{1}{1-z}\sin\frac{1}{z}$$

$$= -(1+z+z^2+\ldots)\left(\frac{1}{z}-\frac{1}{3!\,z^3}+\frac{1}{5!\,z^5}-\ldots\right)$$

$$= -\left(1-\frac{1}{3!}+\frac{1}{5!}-\ldots\right)\frac{1}{z}+\frac{c_{-2}}{z^2}+\frac{c_{-3}}{z^3}+\ldots+\text{regular part},$$

$$c_{-k} \neq 0, \quad k = 2, 3, \ldots.$$

Since the Laurent expansion for this case has an infinite number of terms with negative powers of z, point $z = 0$ is an essential singularity for the integrand. The residue of the integrand at this point is

$$\operatorname*{Res}_{z=0}\frac{\sin\frac{1}{z}}{z-1} = c_{-1} = -\left(1-\frac{1}{3!}+\frac{1}{5!}-\ldots\right) = -\sin 1.$$

Consequently,

$$\int_{|z|=2} \frac{1}{z-1} \sin \frac{1}{z}\, dz = 2\pi i (\sin 1 - \sin 1) = 0.$$

Evaluate the following integrals:

337. $\int_{|z|=1} z \tan \pi z\, dz.$

338. $\int_C \frac{z\, dz}{(z-1)^2 (z+2)}$, where C: $x^{2/3} + y^{2/3} = 3^{2/3}$.

339. $\int_{|z|=2} \frac{e^z\, dz}{z^3 (z+1)}.$

340. $\int_{|z-i|=3} \frac{e^{z^2} - 1}{z^3 - iz^2}\, dz.$

341. $\int_{|z|=1/2} z^2 \sin \frac{1}{z}\, dz.$

342. $\int_{|z|=\sqrt{3}} \frac{\sin \pi z}{z^2 - z}\, dz.$

343. $\int_{|z+1|=4} \frac{z\, dz}{e^z + 3}.$

344. $\int_{|z|=1} \frac{z^2\, dz}{\sin^3 z \cos z}.$

345. $\int_{|z-i|=1} \frac{e^z\, dz}{z^4 + 2z^2 + 1}.$

346. $\int_{|z|=4} \frac{e^{iz}\, dz}{(z-\pi)^3}.$

347. $\int_C \frac{\cos \frac{z}{2}}{z^2 - 4}\, dz$, C: $\frac{x^2}{9} + \frac{y^2}{4} = 1.$

348. $\int \frac{e^{2x}}{z^3 - 1}\, dz$, C: $x^2 + y^2 - 2x = 0.$

349. $\int\limits_C \frac{\sin \pi z}{(z^2-1)^2}\,dz$, C: $\frac{x^2}{4}+y^2=1$.

350. $\int\limits_C \frac{z+1}{z^2+2z-3}\,dz$, C: $x^2+y^2=16$.

351. $\int\limits_C \frac{z\sin z}{(z-1)^5}\,dz$, C: $\frac{x^2}{3}+\frac{y^2}{9}=1$.

352. $\int\limits_C \frac{dz}{z^4+1}$, C: $x^2+y^2=2x$.

353. $\int\limits_{|z|=1} z^3 \sin\frac{1}{z}\,dz$.

354. $\int\limits_{|z|=1/3} (z+1)\,e^{1/z}\,dz$.

355. $\int\limits_{|z|=2/3} \left(\sin\frac{1}{z^2}+e^{z^2}\cos z\right)dz$.

The residue at the point at infinity

We say that a function $f(z)$ is *analytic at the point at infinity* $z=\infty$ if

$$\varphi(\zeta)=f\left(\frac{1}{\zeta}\right)$$

is analytic at point $\zeta=0$. For instance, the function $f(z)=\sin(1/z)$ is analytic at point $z=\infty$ since

$$\varphi(\zeta)=f\left(\frac{1}{\zeta}\right)=\sin\zeta$$

is analytic at point $\zeta=0$.

Suppose a function $f(z)$ is analytic in a neighborhood of the point at infinity (except at point $z=\infty$). Point $z=\infty$ is said to be an *isolated singular point* of $f(z)$ if there are no other singular points in the neighborhood of $z=\infty$. An example of a function that has a nonisolated singular point at the point at infinity is $f(z)=1/\sin z$: the poles $z_k=k\pi$ accumulate at infinity as $k\to\infty$.

The point $z=\infty$ is said to be a *removable singular point*, a *pole*, or an *essential singularity* of a function $f(z)$ depending

on whether $\lim_{z\to\infty} f(z)$ is finite, infinite, or does not exist. The criteria that determine the type of singular point at infinity and are connected with the Laurent expansion are modified in relation to the criteria for a finite singular point.

Theorem 1 *If $z = \infty$ is a removable singular point for a function $f(z)$, the Laurent expansion of $f(z)$ in the neighborhood of this point does not contain terms with positive powers of z; if $z = \infty$ is a pole, the Laurent expansion contains a finite number of terms with positive powers of z; finally, if $z = \infty$ is an essential singularity, the Laurent expansion has an infinite number of terms with positive powers of z.*

The Laurent expansion of a function in the neighborhood of the point at infinity is the expansion of $f(z)$ in a Laurent series that converges everywhere in the exterior of a circle of a sufficiently large radius R and centered at point $z = 0$ (i.e. everywhere except, perhaps, at point $z = \infty$).

Suppose a function $f(z)$ is analytic in a neighborhood of point $z = \infty$ (except, perhaps, at $z = \infty$). We call the *residue of $f(z)$ at the point at infinity* the quantity

$$\operatorname{Res} f(\infty) = \frac{1}{2\pi i}\int_{\gamma^-} f(z)\,dz, \tag{10.1}$$

where γ^- is a sufficiently large circle $|z| = \rho$ traversed clockwise (so that the neighborhood of point $z = \infty$ is to the left, as in the case of a finite point $z = a$). The above definition implies that the value of the residue at infinity is the coefficient of z^{-1} in the Laurent expansion of $f(z)$ about point $z = \infty$ multiplied by -1:

$$\operatorname{Res} f(\infty) = -c_{-1}. \tag{10.2}$$

Example 5. For the function $f(z) = (z + 1)/z$ we have $f(z) = 1 + 1/z$. This expression can be taken as the Laurent expansion of $f(z)$ in the neighborhood of the point at infinity. Obviously,

$$\lim_{z\to\infty} f(z) = 1,$$

so that point $z = \infty$ is a removable singular point and we assume, as usual, that $f(\infty) = 1$. Here $c_{-1} = 1$ and hence,

$$\operatorname{Res} f(\infty) = -1.$$

This example shows that the residue of an analytic function at the point of infinity that is a removable singular point for the function may prove to be nonzero, in contrast to the case of a finite removable singular point.

The well-known expansions of e^z, $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ can be taken as the Laurent expansions about $z = \infty$. Since all these expansions contain an infinite number of terms with positive powers of z, the functions have an essential singularity at $z = \infty$.

Theorem 2 *If a function $f(z)$ has a finite number of singular points in the extended complex plane, the sum of all its residues, including the residue at the point at infinity, is zero.*

This means that if $a_1, a_2, \ldots, a_n$ are the finite singular points of a function $f(z)$, then

$$\operatorname{Res} f(\infty) + \sum_{k=1}^{n} \operatorname{Res} f(a_k) = 0,$$

or

$$\operatorname{Res} f(\infty) = -\sum_{k=1}^{n} \operatorname{Res} f(a_k). \tag{10.3}$$

This relationship proves to be useful in evaluating certain integrals.

Example 6. Evaluate the integral $I = \int\limits_{|z|=2} \frac{dz}{1+z^4}$.

Solution. The roots z_1, z_2, z_3, and z_4 of the equation $z^4 = -1$, which all lie inside the circle $|z| = 2$, are the (finite) poles of the integrand $f(z) = 1/(1 + z^4)$. In the neighborhood of the point at infinity the function $f(z) = 1/(1 + z^4)$ has the following expansion:

$$f(z) = \frac{1}{1+z^4} = \frac{1}{z^4} \frac{1}{1+\frac{1}{z^4}} = \frac{1}{z^4} - \frac{1}{z^8} + \frac{1}{z^{12}} - \ldots,$$

which shows that $\operatorname{Res} f(\infty) = -c_{-1} = 0$. In view of (10.3) we have

$$I = 2\pi i \sum_{k=1}^{4} \operatorname{Res} f(z_k) = -2\pi i \operatorname{Res} f(\infty) = 0.$$

Example 7. Evaluate the integral $I = \int\limits_{|z|=3} \frac{z^{17}}{(z^2+2)^3 (z+3)^4}\, dz$.

Solution. The integrand

$$f(z)=\frac{z^{17}}{(z^2+2)^3(z^3+3)^4}$$

has five singular points inside the circle $|z|=3$ that are multiple poles. Applying the residue theorem leads to cumbersome calculations. Instead it is more convenient to employ the property (10.3), in view of which we have

$$I=-2\pi i \operatorname{Res} f(\infty). \qquad (10.3')$$

Since we can represent $f(z)$ in the form

$$f(z)=\frac{z^{17}}{(z^2+2)^3(z^3+3)^4}$$

$$=\frac{z^{17}}{z^6\left(1+\frac{2}{z^2}\right)^3\left(1+\frac{3}{z^3}\right)^4 z^{12}}=\frac{1}{z}\cdot\frac{1}{\left(1+\frac{2}{z^2}\right)^3\left(1+\frac{3}{z^3}\right)^4},$$

we see that the regular part of the Laurent expansion of this function in the neighborhood of the point at infinity, $z=\infty$, begins with the term $1/z$. Hence, $\operatorname{Res} f(\infty)=-1$. Substituting this value into (10.3), we obtain $I=2\pi i$.

Determine the type of the point at infinity for the following functions:

356. $f(z)=\dfrac{z^3-z^2+z+6}{z^2}$. 357. $f(z)=\dfrac{z+1}{z^1}$.

358. $f(z)=\dfrac{e^z}{z^2}$. 359. $f(z)=\cos\dfrac{1}{z}$.

360. $f(z)=e^{1/z^2}$. 361. $f(z)=z^3e^{1/z}$.

362. Suppose a function $f(z)$ can be represented thus: $f(z)=\varphi(1/z)$, where $\varphi(\zeta)$ is analytic at point $\zeta=0$. Prove that $\operatorname{Res}\limits_{z=\infty} f(z)=-\varphi'(0)$.

Using the notion of the residue at the point at infinity, evaluate the following integrals:

363. $\displaystyle\int_{|z|=1}\frac{z^2+1}{z^3}\,dz$. 364. $\displaystyle\int_{|z|=2}\frac{dz}{1+z^{12}}$.

365. $\int\limits_{|z|=2} \frac{1000z+2}{1+z^{1224}}\,dz$. 366. $\int\limits_{|z|=3} \frac{e^z}{z-1}\,dz$.

367. $\int\limits_{|z|=1} z^3 \sin\frac{1}{z}\,dz$.

368. $\int\limits_{|z|=3} \frac{z^9}{z^{10}-1}\,dz$.

Applying residues to evaluate definite integrals

1. *Integrals of rational functions.* Suppose $f(x)$ is a rational function, i.e. $f(x) = P_m(x)/Q_n(x)$, where $P_m(x)$ and $Q_n(x)$ are polynomials of degrees m and n, respectively. If $f(x)$ is continuous on the entire real axis ($Q_n(x) \neq 0$) and $n \geqslant m + 2$, i.e. the degree of the denominator is greater than that of the numerator by at least two units, then

$$\int\limits_{-\infty}^{+\infty} f(x)\,dx = 2\pi i\sigma, \tag{10.4}$$

where σ stands for the sum of the residues of the function $f(z) = P_m(z)/Q_n(z)$ at all the poles that lie in the upper half-plane.

Example 8. Evaluate the integral $I = \int\limits_0^{\infty} \frac{x^2\,dx}{(x^2+a^2)^2}$ $(a > 0)$.

Solution. Since the integrand $f(x) = x^2/(x^2 + a^2)^2$ is an even function $I = \frac{1}{2}\int\limits_{-\infty}^{+\infty} \frac{x^2\,dx}{(x^2+a^2)^2}$. We introduce the function $f(z) = z^2/(z^2 + a^2)^2$, which coincides with $f(x)$ on the real axis, i.e. when $z = x$. This function has a second-order pole in the upper half-plane at $z = ai$. The residue of $f(z)$ at this pole is

$$\operatorname{Res} f(ai) = \lim_{z\to ai} \frac{d}{dz}\left[f(z)(z-ai)^2\right]$$

$$= \lim_{z\to ai} \frac{d}{dz}\left[\frac{z^2}{(z+ai)^2}\right] = \lim_{z\to ai} \frac{2aiz}{(z+ai)^3} = \frac{1}{4ai}.$$

Employing (10.4) yields

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2\,dx}{(x^2+a^2)^2} = \frac{1}{2}\cdot 2\pi i\cdot\frac{1}{4ai} = \frac{\pi}{4a}.$$

Evaluate the following integrals with infinite limits:

369. $\int_0^{\infty} \frac{x^2+1}{x^4+1}\,dx.$

370. $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$ $(a>0,\ b>0).$

371. $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^3}.$ 372. $\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^{n+1}}.$

373. $\int_{-\infty}^{+\infty} \frac{x\,dx}{(x^2+4x+13)^2}.$ 374. $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)^2(x^2+b^2)^2}.$

375. $\int_0^{+\infty} \frac{x^4+1}{x^6+1}\,dx.$ 376. $\int_{-\infty}^{+\infty} \frac{x^{2m}}{1+x^{2n}}\,dx.$

377. $\int_{-\infty}^{+\infty} \frac{dx}{1+x^6}.$ 378. $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+2x+2)^3}.$

379. $\int_{-\infty}^{+\infty} \frac{x^4\,dx}{(a+bx^2)^4}$ $(a>0,\ b>0).$

380. Prove that $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}\pi.$

2. *Integrals of the type*

$$\int_0^{\infty} R(x)\cos\lambda x\,dx, \quad \int_0^{\infty} R(x)\sin\lambda x\,dx,$$

where $R(x)$ is a proper rational fraction, and λ is a positive integer. In evaluating such integrals it is convenient to use

Jordan's lemma *Suppose $g(z)$ is a function that is analytic in the upper half-plane $(0 < \arg z < \pi)$ except, perhaps, at*

a finite number of singular points, and tends to zero as $|z| \to \infty$ *in the half-plane. Then for* $\lambda > 0$ *we have*

$$\lim_{R\to\infty} \int_{C_R} g(z)\, e^{i\lambda z}\, dz = 0,$$

where the curve C_R *is a semicircle in the upper half-plane with a radius* R *and centered at point* 0 (Fig. 7).

Example 9. Evaluate the integral $I = \int_0^\infty \frac{x \sin ax}{x^2+k^2}\, dx$ $(a>0,\ k>0)$.

Solution. We introduce the auxiliary function $f(z) = \frac{ze^{iaz}}{z^2+k^2}$. We can easily see that $\operatorname{Im} f(x)$ coincides with

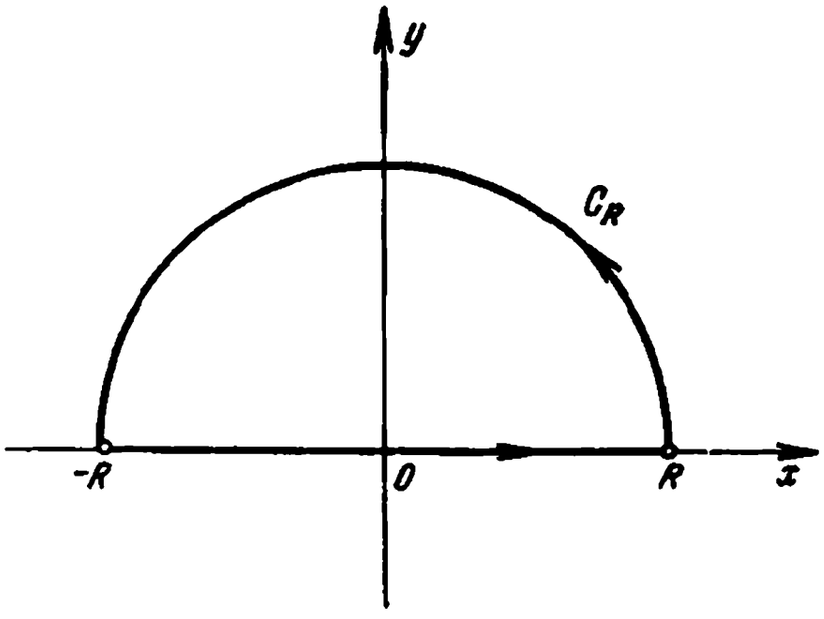


Figure 7

the integrand $\varphi(x) = \frac{x \sin ax}{x^2+k^2}$ when $z = x$. Now we consider the closed curve in Fig. 7. For sufficiently large R's the function $g(z) = z/(z^2 + k^2)$ satisfies on C_R the following inequality: $|g(z)| < R/(R^2 - k^2)$; hence, $g(z)$ tends to zero as $R \to \infty$. By Jordan's lemma,

$$\lim_{R\to\infty} \int_{C_R} \frac{ze^{iaz}}{z^2+k^2}\, dz = 0. \qquad (10.5)$$

For all $R > k$ the residue theorem states that

$$\int_{-R}^{R} \frac{xe^{iax}}{x^2+k^2}\,dx + \int_{C_R} \frac{ze^{iaz}}{z^2+k^2}\,dz = 2\pi i\sigma,$$

where

$$\sigma = \operatorname*{Res}_{z=ik}\left[\frac{ze^{iaz}}{z^2+k^2}\right] = \lim_{z\to ik}\left[\frac{ze^{iaz}}{z^2+k^2}(z-ik)\right] = \frac{1}{2}e^{-ak}.$$

In the limit as $R \to \infty$ we have, allowing for (10.5),

$$\int_{-\infty}^{+\infty} \frac{xe^{iax}}{x^2+k^2}\,dx = \pi i e^{-ak}.$$

Separating the real and imaginary parts, we find that

$$\int_{-\infty}^{+\infty} \frac{x\sin ax}{x^2+k^2}\,dx = \pi e^{-ak}.$$

In view of the fact that the integrand is an even function, we finally see that

$$I = \frac{\pi}{2}e^{-ak}.$$

Example 10. Find the integral representation of the unit-step function (Heaviside's unit function), which is the real function $f(t)$ whose value is 0 if t is negative and whose value is 1 if t is positive.

Solution. We take the function

$$f(t) = \frac{1}{2\pi i}\int_C \frac{e^{-izt}}{z}\,dz,$$

where the curve C is shown in Fig. 8. Closing this curve by the semicircle C_R in the upper half-plane, we note that for negative t's, due to Jordan's lemma,

$$\int_{C_R} \frac{e^{-izt}}{z}\,dz \to 0 \quad \text{as} \quad R \to \infty,$$

and since in the domain bounded by such a closed curve the integrand is analytic, $f(t) = 0$ when $t < 0$. Now we close C by the semicircle C_R' in the lower half-plane. In view of

Jordan's lemma, for $t > 0$ we find that

$$\int_{C_R^1} \frac{e^{-izt}}{z}\,dz \to 0 \quad \text{as} \quad R \to \infty.$$

But now point $z = 0$ lies inside the integration contour. In view of the residue theorem this means that

$$f(t) = \frac{1}{2\pi i}\int_C \frac{e^{-izt}}{z}\,dz = 1 \quad (t > 0).$$

Thus,

$$f(t) = \frac{1}{2\pi i}\int_C \frac{e^{-izt}}{z}\,dz = \begin{cases} 0 & \text{at} \quad t < 0, \\ 1 & \text{at} \quad t > 0. \end{cases}$$

Therefore, the above integral represents a discontinuous function.

Evaluate the following integrals:

381. $\int_{-\infty}^{+\infty} \frac{x\cos x\,dx}{x^2-2x+10}$. 382. $\int_{-\infty}^{+\infty} \frac{x\sin x}{x^2+4x+20}\,dx$.

383. $\int_0^{\infty} \frac{\cos x\,dx}{(x^2+1)(x^2+4)}$. 384. $\int_{-\infty}^{+\infty} \frac{\cos x\,dx}{x^2+9}$.

385. $\int_0^{\infty} \frac{\cos ax}{1+x^4}\,dx \quad (a > 0)$. 386. $\int_0^{\infty} \frac{\cos x}{x^2+a^2}\,dx \quad (a > 0)$.

387. $\int_0^{\infty} \frac{\cos mx}{a^2+x^2}\,dx \quad (m > 0,\ a > 0)$. 388. $\int_0^{\infty} \frac{x\sin x}{1+x^2+x^4}\,dx$.

389. $\int_{-\infty}^{+\infty} \frac{\cos \lambda x}{(x^2+1)(x^2+9)} \quad (\lambda > 0)$.

390. $\int_0^{\infty} \frac{x^3\sin ax}{(1+x^2)^2}\,dx \quad (a > 0)$. 391. $\int_0^{\infty} \frac{x^2\cos x\,dx}{(x^2+1)^2}$.

392. $\int_0^{\infty} \frac{3x^2-a^2}{(x^2+b^2)^2}\cos mx\,dx$.

Example 11. Evaluate the integral

$$I=\int_0^{\infty}\frac{\sin ax}{x(x^2+b^2)}\,dx \quad (a>0,\ b>0). \tag{10.6}$$

Solution. We introduce the function $f(z)=\dfrac{e^{iaz}}{z(z^2+b^2)}$, whose imaginary part coincides with the integrand in (10.6) when $z=x$. It has a singularity on the real axis, a first-

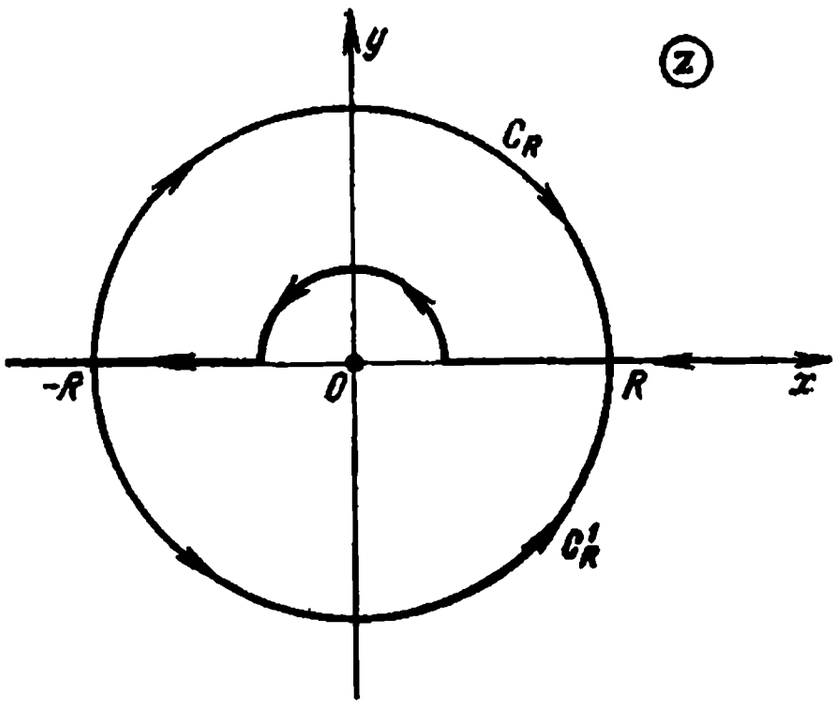


Figure 8

order pole at point $z=0$. For this reason we choose the integration contour as shown in Fig. 9 (the singular point $z=0$ is circuited by a small semicircle C_r $(r<b)$, and the semicircle C_R is selected in such a way that $b<R$). Thus, inside the closed contour there is only one pole of $f(z)$, namely, at point $z=ib$. According to the residue theorem,

$$\int_{-R}^{-r}\frac{e^{iax}}{x(x^2+b^2)}\,dx+\int_{C_r}\frac{e^{iaz}}{z(z^2+b^2)}\,dz+\int_{r}^{R}\frac{e^{iax}\,dx}{x(x^2+b^2)}$$

$$+\int_{C_R}\frac{e^{iaz}}{z(z^2+b^2)}\,dz=2\pi i\sigma, \tag{10.7}$$

where

$$\sigma = \operatorname*{Res}_{z=ib} \frac{e^{iaz}}{z(z^2+b^2)} = \lim_{z\to bi} \frac{e^{iaz}(z-ib)}{z(z^2+b^2)} = -\frac{e^{-ab}}{2b^2}. \qquad (10.8)$$

Substituting $-x$ for x in the first integral on the left-hand

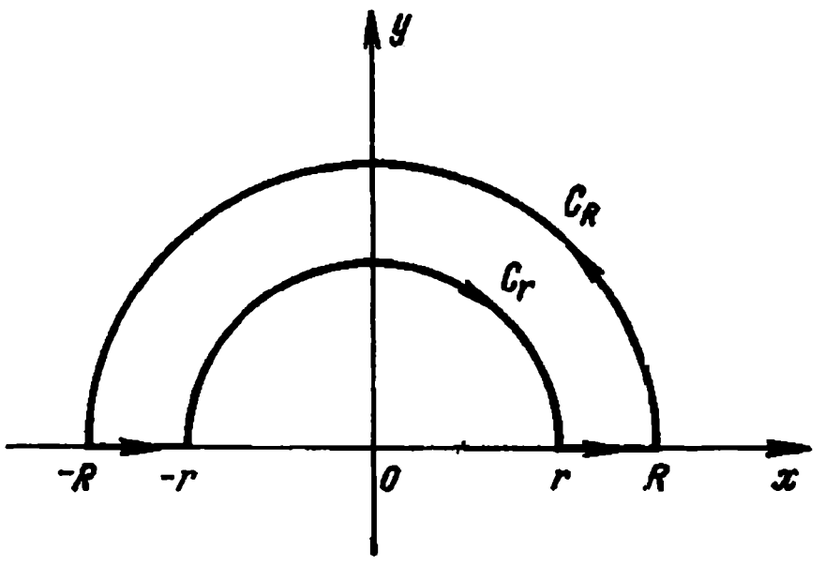


Figure 9

side of (10.7) and combining it with the third integral, we obtain

$$\int_{-R}^{-r} \frac{e^{iax}}{x(x^2+b^2)}\,dx + \int_{r}^{R} \frac{e^{iax}}{x(x^2+b^2)}\,dx = \int_{r}^{R} \frac{e^{iax}-e^{-iax}}{x(x^2+b^2)}\,dx$$

$$= 2i\int_{r}^{R} \frac{\sin ax\,dx}{x(x^2+b^2)}. \qquad (10.9)$$

Since

$$\lim_{z\to 0} \frac{e^{iaz}}{z^2+b^2} = \frac{1}{b^2},$$

we can write the integrand $\dfrac{e^{iaz}}{z(z^2+b^2)}$ in the following form:

$$\frac{e^{iaz}}{z(z^2+b^2)} = \frac{1}{b^2}\,\frac{1}{z} + \frac{\Psi(z)}{z},$$

with $\psi(z)$ tending to zero as $z \to 0$. Putting $z = re^{i\varphi}$, we find that

$$\int_{C_r} \frac{e^{iaz}\,dz}{z(z^2+b^2)} = \frac{1}{b^2}\int_{C_r} \frac{dz}{z} + \int_{C_r} \frac{\psi(z)}{z}\,dz$$

$$= -\frac{i\pi}{b^2} + i\int_{\pi}^{0} \psi(re^{i\varphi})\,d\varphi. \qquad (10.10)$$

As $r \to 0$, the integral on the right-hand side of (10.10) tends to zero:

$$\lim_{r\to 0} \int_{\pi}^{0} \psi(re^{i\varphi})\,d\varphi = 0. \qquad (10.11)$$

Finally, according to Jordan's lemma, the fourth integral on the left-hand side of (10.7) tends to zero as $R \to \infty$, since the function $g(z) = \dfrac{1}{z(z^2+b^2)}$ tends to zero as $|z| \to \infty$:

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iaz}}{z(z^2+b^2)}\,dz = 0. \qquad (10.12)$$

Thus, as $R \to \infty$ and $r \to 0$ we have for (10.7), allowing for (10.8)-(10.12),

$$2i\int_0^{\infty} \frac{\sin ax}{x(x^2+b^2)}\,dx - \frac{\pi i}{b^2} = -\pi i\,\frac{e^{-ab}}{b^2};$$

whence

$$\int_0^{\infty} \frac{\sin ax}{x(x^2+b^2)}\,dx = \frac{\pi}{2b^2}(1-e^{-ab}).$$

Evaluate the following integrals

393. $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\,dx$. **394.** $\displaystyle\int_0^{\infty} \frac{\sin x}{x(x^2+1)}\,dx$.

395. $\displaystyle\int_0^{\infty} \frac{\cos ax - \cos bx}{x^2}\,dx \quad (a>0,\ b>0)$.

396. $\displaystyle\int_0^{\infty} \frac{\sin mx}{x(x^2+a^2)^2}\,dx$.

3. *Evaluating integrals with the exponential function.*
Example 12. Evaluate Fresnel's integrals

$$I_1 = \int_0^\infty \cos x^2\, dx, \quad I_2 = \int_0^\infty \sin x^2\, dx,$$

knowing that

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}. \tag{10.13}$$

Solution. We introduce the auxiliary function $f(z) = e^{iz^2}$ and an integration contour shown in Fig. 10 (the circular

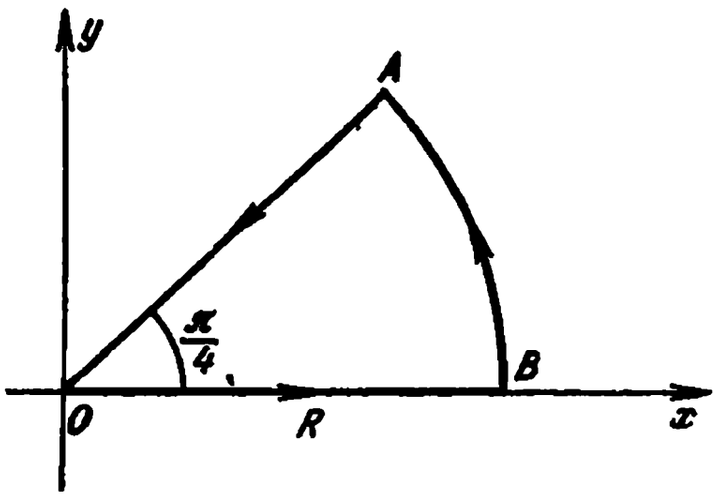


Figure 10

sector $OBAO$, where $OA = OB = R$ and $\angle BOA = \pi/4$). Inside this contour the function $f(z)$ is analytic, and by the residue theorem

$$\int_{OBAO} e^{iz^2}\, dz = \int_0^R e^{ix^2}\, dx + \int_{C_R} e^{iz^2}\, dz + \int_{AO} e^{iz^2}\, dz = 0. \tag{10.14}$$

We will now show that

$$\lim_{R\to\infty} \int_{C_R} e^{iz^2}\, dz = 0. \tag{10.15}$$

Indeed, putting $z^2 = \xi$, we obtain $dz = d\xi/2\sqrt{\xi}$ and

$$\int_{C_R} e^{iz^2}\, dz = \int_{\Gamma_{R^2}} \frac{e^{i\xi}}{2\sqrt{\xi}}\, d\xi,$$

where Γ_{R^2} is a quarter of the circumference of a circle of radius R^2. The function $g(\xi) = 1/2\sqrt{\xi}$ satisfies the hypothesis of Jordan's lemma, which means that

$$\lim_{R\to\infty} \int_{\Gamma_{R^2}} \frac{e^{i\xi}}{2\sqrt{\xi}}\, d\xi = \lim_{R\to\infty} \int_{C_R} e^{iz^2}\, dz = 0.$$

On the segment AO we have $z = \rho e^{i\pi/4}$ and $z^2 = \rho^2 e^{i\pi/2} = \rho^2 i$ $(0 \leqslant \rho \leqslant R)$. Whence

$$\int_{AO} e^{iz^2}\, dz = \int_R^0 e^{-\rho^2} e^{i\frac{\pi}{4}}\, d\rho = -e^{i\frac{\pi}{4}} \int_0^R e^{-\rho^2}\, d\rho. \quad (10.16)$$

Going over to the limit in (10.14) as $R \to \infty$ and allowing for (10.15), (10.16), and (10.13), we obtain

$$\int_0^\infty e^{ix^2}\, dx = e^{i\frac{\pi}{4}} \frac{\sqrt{\pi}}{2},$$

or

$$\int_0^\infty \cos x^2\, dx + i \int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} + i\frac{1}{2}\sqrt{\frac{\pi}{2}},$$

from which we find that

$$\int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}, \quad \int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Example 13. Evaluate the integral $\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^x}\, dx$ $(0 < a < 1)$.

Solution. We introduce the auxiliary function $f(z) = e^{az}/(1 + e^z)$ and an integration contour shown in Fig. 11 (a rectangle with sides $2R$ and 2π). Inside this contour the function $f(z)$ is analytic except at point $z = \pi i$, which in this case is a simple pole with

$$\operatorname{Res} f(\pi i) = \frac{e^{az}}{(1+e^z)'}\Big|_{z=\pi i} = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i}.$$

By the residue theorem we have

$$\int_{DA} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz = 2\pi i e^{a\pi i}. \tag{10.17}$$

On the segment DA we have $z = x$ $(-R \leqslant x \leqslant R)$, whence

$$\int_{DA} = \int_{-R}^{R} \frac{e^{ax}}{1+e^x}\,dx. \tag{10.18}$$

On the segment AB we have $z = R + iy$ $(0 \leqslant y \leqslant 2\pi)$

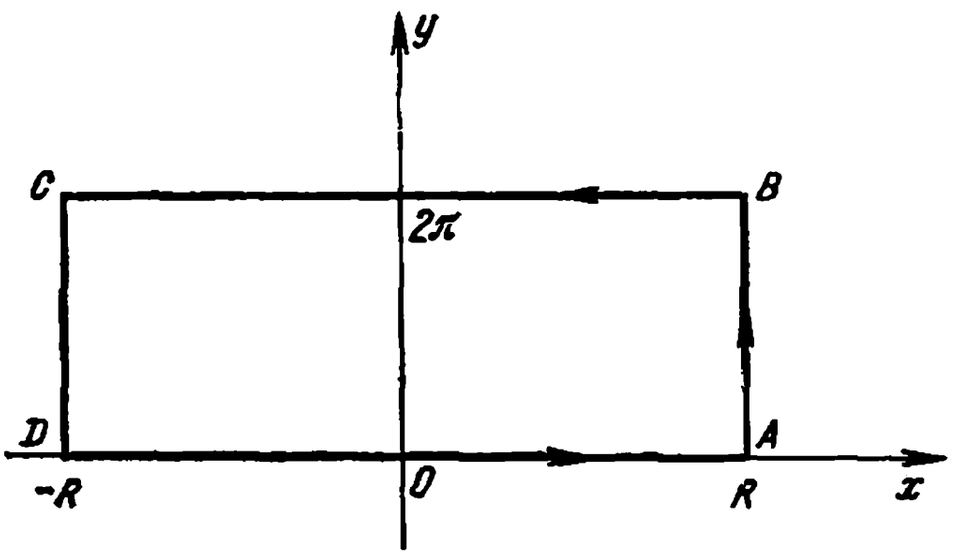


Figure 11

whence

$$\left|\frac{e^{az}}{1+e^z}\right| = \left|\frac{e^{a(R+iy)}}{1+e^{R+iy}}\right| \leqslant \frac{e^{aR}}{e^R-1}.$$

This means that

$$\left|\int_{AB}\right| \leqslant \int_0^{2\pi} \frac{e^{aR}}{e^R-1}\,dy = \frac{2\pi e^{aR}}{e^R-1} \to 0 \quad \text{as } R \to \infty$$

$$\text{(since } 0 < a < 1\text{).} \tag{10.19}$$

Similarly,

$$\left|\int_{CD} f(z)\,dz\right| \leqslant \frac{e^{-aR}}{1-e^{-R}}\,2\pi \to 0 \quad \text{as } R \to \infty. \tag{10.20}$$

On the segment BC we have $z = x + 2\pi i$ $(-R \leqslant x \leqslant R)$, whence

$$\int_{BC} = \int_{R}^{-R} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\,dx = -e^{2a\pi i}\int_{-R}^{R} \frac{e^{ax}}{1+e^{x}}\,dx. \qquad (10.21)$$

Going over to the limit in (10.17) as $R \to \infty$ and allowing for (10.18)-(10.21), we find that

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^{x}}\,dx - e^{2a\pi i}\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^{x}}\,dx = -2\pi i e^{a\pi i},$$

whence

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{2\pi i e^{a\pi i}}{e^{2a\pi i}-1} = \frac{\pi}{\sin a\pi}.$$

Evaluate the following integrals that contain the exponential function:

397. $\int_{0}^{\infty} e^{-ax^2}\cos bx\,dx$ $(a > 0,\ b > 0)$.

Hint. Take $f(z) = e^{-az^2}$ and a contour in the form of a rectangle with sides $2R$ and $b/2a$.

398. $\int_{-\infty}^{+\infty} \frac{e^{ax}-e^{bx}}{1+e^{x}}\,dx$ $(0<a<1,\ 0<b<1)$.

Hint. Take $f(z) = \frac{e^{az}-e^{bz}}{1+e^{z}}$ and the contour shown in Fig. 11.

4. *Evaluating integrals of the type*

$$\int_{0}^{2\pi} R(\cos x, \sin x)\,dx, \qquad (10.22)$$

where R is a rational function of $\cos x$ and $\sin x$ that is bounded in the interval of integration. We put $e^{ix} = z$. Then $dx = dz/iz$ and

$$\cos x = \frac{z^2+1}{2z}, \quad \sin x = \frac{z^2-1}{2iz}.$$

Here, obviously, $|z| = 1$ and $0 \leqslant x \leqslant 2\pi$. Integral (10.22) then takes the form

$$\int_C F(z)\,dz, \tag{10.23}$$

where C is a circle of unit radius centered at the origin of coordinates. According to the residue theorem, the value of integral (10.23) is $2\pi\sigma i$, where σ stands for the sum of the residues at the poles that lie inside circle C.

Example 14. Evaluate the integral $I = \int_0^{2\pi} \frac{dx}{(a+b\cos x)^2}$ $(a > b > 0)$.

Solution. Using the substitution $e^{ix} = z$, we obtain after simple transformations

$$I = \frac{4}{i}\int_C \frac{z\,dz}{(bz^2+2az+b)^2} = \frac{4}{i}2\pi i\sum_{k=1}^{n} \operatorname{Res} F(z_k).$$

Inside the unit circle, provided $a > b > 0$, there is only one pole (of order 2)

$$z_1 = \frac{-a+\sqrt{a^2-b^2}}{b}.$$

The residue of $F(z) = \frac{z}{(bz^2+2az+b)^2}$ at this pole is

$$\operatorname{Res} F(z_1) = \lim_{z\to z_1}\frac{d}{dz}\left\{\frac{z(z-z_1)^2}{b^2(z-z_1)^2(z-z_2)^2}\right\} = \frac{a}{4}(a^2-b^2)^{-3/2}.$$

Thus,

$$I = \frac{2\pi a}{(a^2-b^2)^{3/2}}.$$

Evaluate the following integrals:

399. $\int_0^{2\pi} \frac{dx}{1-2p\cos x+p^2}$ $(0<p<1)$.

400. $\int_0^{2\pi} \frac{\cos^2 3x\,dx}{1-2p\cos 2x+p^2}$ $(0<p<1)$.

401. $\int_0^{2\pi} \frac{\cos 2x\,dx}{1-2p\cos x+p^2}\ (p>1)$.

402. $\int_0^{2\pi} \frac{\cos x\,dx}{1-2p\sin x+p^2}\ (0<p<1)$.

403. $\int_0^{2\pi} \frac{dx}{a+\cos x}\ (a>1)$.

404. $\int_0^{\pi} \cot(x-a)\,dx \quad (\operatorname{Im} a>0)$.

405. $\int_0^{2\pi} \frac{\sin^2 x}{a+b\cos x}\,dx \quad (a>b>0)$.

406. $\int_0^{2\pi} \frac{dx}{1+a\cos x}\ (0<a<1)$.

407. $\int_0^{2\pi} \frac{dx}{a+b\cos x} \quad (0<b<a)$.

Summation of series by means of residues

1. Let a function $f(z)$ be analytic in the entire complex plane except at a finite number of poles $z_1, z_2, \ldots, z_k$ that do not coincide with any of the points $z = 0, \pm 1, \pm 2, \ldots$, and let $f(z)$ satisfy the condition $f(z) = O(z^{-2})$ as $z \to \infty$. (The notation $f(z) = O(g(z))$ as $z \to \infty$ means that the quotient $f(z)/g(z)$ remains bounded as $z \to \infty$, i.e. $|f(z)/g(z)| \leqslant C$, where C is a positive constant.) Then

$$\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{m=1}^{k} \operatorname*{Res}_{z=z_m} [f(z)\cot \pi z]. \qquad (10.24)$$

Example 15. Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+a^2}$, where a is nonzero.

Solution. We take the function $f(z) = 1/(z^2 + a^2)$, which is analytic everywhere except at points $z_1 = ai$ and $z_2 = -ai$, which are simple poles. Since

$$f(z) = \frac{1}{z^2\left(1 + \frac{a^2}{z^2}\right)},$$

it follows that

$$\frac{1}{z^2+a^2} = O(z^{-2}) \quad \text{as} \quad z \to \infty.$$

Employing formula (10.24), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = -\pi\left[\operatorname*{Res}_{z=ai} \frac{\cot \pi z}{z^2+a^2} + \operatorname*{Res}_{z=-ai} \frac{\cot \pi z}{z^2+a^2}\right].$$

Points $z_1 = ai$ and $z_2 = -ai$ are simple poles of the function $\frac{\cot \pi z}{z^2+a^2}$, which means that its residues are

$$\operatorname*{Res}_{z=z_k} \frac{\cot \pi z}{z^2+a^2} = \frac{\cot \pi z}{2z}\Big|_{z=z_k} = \frac{\cot \pi z_k}{2z_k}, \quad k = 1, 2.$$

Then

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = -\pi\left(\frac{\cot \pi a i}{2ai} + \frac{\cot(-\pi a i)}{-2ai}\right)$$

$$= \frac{\pi}{a} i \cot(\pi a i) = \frac{\pi}{a} \coth \pi a.$$

We can write the series on the left-hand side as

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \cdots + \frac{1}{(-n)^2+a^2} + \cdots + \frac{1}{(-2)^2+a^2}$$

$$+ \frac{1}{(-1)^2+a^2} + \frac{1}{0^2+a^2} + \frac{1}{1^2+a^2}$$

$$+ \frac{1}{2^2+a^2} + \cdots + \frac{1}{n^2+a^2} + \cdots$$

$$= \frac{1}{a^2} + 2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2}.$$

From this we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} - \frac{1}{2a^2}.$$

The sought-for sum of the given series is

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{2} \frac{\pi}{a} \coth \pi a - \frac{1}{2a^2} = \frac{\pi a \coth \pi a - 1}{2a^2}.$$

Find the sums of the following series with a a noninteger:

408. $\sum_{n=1}^{\infty} \frac{1}{n^2-a^2}$. 409. $\sum_{n=1}^{\infty} \frac{1}{n^4-a^4}$.

410. $\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2}$. 411. $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

2. Let a function $f(z)$ be analytic in the entire complex plane except at a finite number of poles $z_1, z_2, \ldots, z_k$ that do not coincide with any of the points $z = 0, \pm 1, \pm 2, \ldots$ $\ldots$, and let $f(z)$ satisfy the inequality

$$|f(z)| \leqslant e^{a|\operatorname{Im} z|} \varepsilon(|z|), \qquad (10.25)$$

where $\varepsilon(|z|) \to 0$ as $z \to \infty$, $z \in G_\rho$; $0 \leqslant a \leqslant \pi$. Here G_ρ is the entire complex plane from which the circles $|z - z_m| \leqslant \rho$, $m = 1, 2, \ldots, k$, are excluded. Then the following formula is valid:

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\pi \sum_{m=1}^{k} \operatorname*{Res}_{z=z_m} \frac{f(z)}{\sin \pi z}. \qquad (10.26)$$

Example 16. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+a^2}$, $a \neq 0$.

Solution. The function $f(z) = 1/(z^2 + a^2)$ has two simple poles, $z_1 = ai$ and $z_2 = -ai$, and satisfies condition (10.25), since

$$|f(z)| = \frac{1}{|z^2+a^2|} \leqslant \frac{1}{|z|^2 - |a|^2},$$

where $\alpha = 0$ and $\varepsilon(|z|) = (|z|^2 - |a|^2)^{-1} \to 0$ as $z \to \infty$. Applying (10.26), we find that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2+a^2}$$

$$= -\pi \left[\operatorname*{Res}_{z=ai} \frac{1}{(z^2+a^2)\sin \pi z} + \operatorname*{Res}_{z=-ai} \frac{1}{(z^2+a^2)\sin \pi z} \right].$$

The residues of $\dfrac{1}{(z^2+a^2)\sin \pi z}$ at points z_k, $k = 1, 2$, are

$$\operatorname*{Res}_{z=z_k} \frac{1}{(z^2+a^2)\sin \pi z} = \frac{1}{2z_k \sin \pi z_k + (z_k^2+a^2)\pi \cos \pi z_k},$$

whence

$$\operatorname*{Res}_{z=ai} \frac{1}{(z^2+a^2)\sin \pi z} = \frac{1}{2ai \sin \pi ai} = -\frac{1}{2a \sinh \pi a},$$

$$\operatorname*{Res}_{z=-ai} \frac{1}{(z^2+a^2)\sin \pi z} = -\frac{1}{2a \sinh \pi a}.$$

Consequently,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2+a^2} = \frac{1}{a \sinh \pi a}.$$

Next we can write

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2+a^2} = \cdots + \frac{(-1)^{-n}}{(-n)^2+a^2} + \cdots + \frac{(-1)^{-2}}{(-2)^2+a^2}$$

$$+ \frac{(-1)^{-1}}{(-1)^2+a^2} + \frac{(-1)^0}{0^2+a^2} + \frac{(-1)^1}{1^2+a^2} + \frac{(-1)^2}{2^2+a^2} + \cdots + \frac{(-1)^n}{n^2+a^2} + \cdots$$

$$= \frac{1}{a^2} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+a^2}.$$

Whence

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+a^2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2+a^2} - \frac{1}{2a^2},$$

which means that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+a^2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2+a^2} + \frac{1}{2a^2}$$

$$= \frac{\pi}{2a \sinh \pi a} + \frac{1}{2a^2} = \frac{1}{2a^2} \left(1 + \frac{\pi a}{\sinh \pi a}\right).$$

Find the sums of the following series with a a noninteger:

412. $\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2}$. 413. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$.

414. $\sum_{n=-\infty}^{\infty} \frac{(-1)^n \cos \alpha n}{n^2+a^2}$, $-\pi < \alpha < \pi$.

415. $\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n^2+a^2)^2}$.

§ 11 The Logarithmic Residue. The Argument Principle. Rouche's Theorem

The *logarithmic derivative* of a function $f(z)$ is a function $\varphi(z)$ that is the derivative of the logarithm of $f(z)$:

$$\varphi(z) = [\operatorname{Ln} f(z)]' = \frac{f'(z)}{f(z)}.$$

Only the zeros and singular points of $f(z)$ can be the singular points of $\varphi(z)$.

The residue of the logarithmic derivative of $f(z)$ at a point that is a zero of $f(z)$ is equal to the order of the zero, while at a point that is a pole of $f(z)$ the residue is equal to the order of the pole multiplied by -1.

Example 1. Find the residues of the logarithmic derivative of the function $f(z) = (\sin z)/(z+1)$ at the zeros and poles of the function.

Solution. The function has an infinite number of simple zeros $z = k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$) and one simple pole $z = -1$. Whence,

$$\operatorname*{Res}_{z=k\pi} \frac{f'(z)}{f(z)} = 1 \quad (k = 0, -1, -2, \ldots), \quad \operatorname*{Res}_{z=-1} \frac{f'(z)}{f(z)} = -1.$$

Find the residues of the logarithmic derivatives of the following functions at their zeros and poles:

416. $f(z)=\frac{\sin z}{z}$. 417. $f(z)=\cos^3 z$.

418. (a) $f(z)=\frac{\cos z}{z}$; (b) $f(z)=\sin z$.

Suppose a function $f(z)\neq 0$ is analytic at all points of a closed curve C. The quantity

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$$

is said to be the *logarithmic residue of $f(z)$ with respect to the closed curve C.*

The logarithmic residue theorem *Suppose we have a function $f(z)$ that is analytic in a closed domain D except at a finite number of poles and that has neither zeros nor poles on the boundary C of D. Then the difference between the number of zeros and that of poles of $f(z)$ in D (both quantities are calculated with due regard for the orders of the poles and zeros) is equal to the logarithmic residue of $f(z)$ with respect to C:*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz=N-P,$$

where N is the number of zeros of $f(z)$ in D, and P the number of poles of $f(z)$ in D.

The logarithmic residue of the polynomial

$$Q_n(z)=\sum_{k=0}^{n} a_k z^k$$

with respect to contour C is equal to the number of zeros of the polynomial (with due regard for the orders of the zeros) in the domain D bounded by C.

Example 2. Find the logarithmic residue of the function $f(z)=(\cosh z)/(e^{iz}-1)$ with respect to C: $|z|=8$.

Solution. First we find the zeros z_k of $f(z)$ by solving the equation $\cosh z=0$, or $e^z+e^{-z}=0$. If we write the latter as $e^{2z}=-1$, we find that $2z=\operatorname{Ln}(-1)=(2k+1)\pi i$, so that $z_k=\frac{2k+1}{2}\pi i$ ($k=0,\pm 1,\pm 2,\ldots$) (all the zeros are simple). Next we find the poles of $f(z)$ by solving the equation $e^{iz}-1=0$, or $e^{iz}=1$. This yields $iz=\operatorname{Ln} 1=2m\pi i$, or $z_m=2m\pi$ ($m=0,\pm 1,\pm 2,\ldots$). The zeros

and simple poles of $f(z)$ that lie in the circle $|z|<8$ are, respectively,

$$z_k=\frac{2k+1}{2}\pi i \quad (k=0, \pm 1, \pm 2, -3)$$

and

$$z_m = 2m\pi \ (m = 0, \pm 1).$$

The number of zeros N is 6 and the number of poles P is 3. Therefore,

$$\frac{1}{2\pi i}\int\limits_{|z|=8}\frac{f'(z)}{f(z)}dz=6-3=3.$$

Example 3. Find the logarithmic residue of the function

$$f(z)=\frac{1+z^2}{1-\cos 2\pi z}$$

with respect to the circle $|z| = \pi$.

Solution. Putting $1 + z^2 = 0$, we find the two (simple) zeros of $f(z)$, namely, $a_1 = -i$ and $a_2 = i$. Putting $1 - \cos 2\pi z = 0$, we find the poles of $f(z)$, namely, $z_n = n$, $n = 0, \pm 1, \pm 2, \ldots$. The order of all the poles is 2. In the circle $|z| < \pi$ the given function has two simple zeros, $a_1 = -i$ and $a_2 = i$, and seven second-order poles,

$$z_1 = -3,\ z_2 = -2,\ z_3 = -1,\ z_4 = 0,\ z_5 = 1,\ z_6 = 2,\ z_7 = 3.$$

Thus, $N = 2$ and $P = 7$. By virtue of the logarithmic residue theorem we find that the logarithmic residue of $f(z)$ with respect to the circle $|z| = \pi$ is

$$\frac{1}{2\pi i}\int\limits_{|z|=\pi}\frac{f'(z)}{f(z)}dz=2-7\times 2=-12.$$

In the problems below find the logarithmic residues of the functions with respect to the contours specified:

419. $f(z)=\frac{z}{1+z^3}$, C: $|z| = 2$.

420. $f(z) = \cos z + \sin z$, C : $|z| = 4$.

421. $f(z) = (e^z - 2)^2$, C : $|z| = 8$.

422. $f(z) = \tanh z$, C : $|z| = 8$.

423. $f(z) = \tan^3 z$, C : $|z| = 6$.

424. $f(z) = 1 - \tanh^2 z$, C : $|z| = 2$.

The argument principle *The logarithmic residue of a function $f(z)$ with respect to a closed curve C is equal to the vario-*

tion $\Delta_C \operatorname{Arg} f(z)$ *as* z *traverses* C *divided by* 2π:

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi}\Delta_C \operatorname{Arg} f(z).$$

Consequently, the difference between the number of zeros and poles of $f(z)$ lying in the domain D bounded by C is

$$N - P = \frac{1}{2\pi}\Delta_C \operatorname{Arg} f(z).$$

In other words, the difference $N - P$ is equal to the number of revolutions about the origin of coordinates that the vector w, which runs from point $w = 0$ to point $w = f(z)$, completes as point z traverses C (the number of revolutions is assumed to be positive if w revolves in the positive sense and negative if otherwise). In the particular case where $w = f(z)$ is analytic in the domain D and on the boundary C of D and has no zeros on C, the logarithmic residue of $f(z)$ with respect to C gives the number of zeros of $f(z)$ that lie in D and is equal to the variation of $\operatorname{Arg} f(z)$ as z traverses C divided by 2π:

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi}\Delta_C \operatorname{Arg} f(z) = N.$$

For instance, this is the case for the polynomial $Q_n(z) = \sum_{k=0}^{n} a_k z^k$.

Example 4. Find the number of roots in the right half-plane $\operatorname{Re} z > 0$ of the equation

$$Q_5(z) \equiv z^5 + z^4 + 2z^3 - 8z - 1 = 0.$$

Solution. In view of the argument principle the number of zeros inside C is

$$N = \frac{1}{2\pi}\Delta_C \operatorname{Arg} Q_5(z),$$

where C consists of the semicircle $C_R : |z| = R$, $\operatorname{Re} z > 0$, and the diameter of C_R along the imaginary axis; we assume R to be so large that all the zeros of $Q_5(z)$ that lie in the right half-plane prove to be inside the semicircle $|z| < R$, $\operatorname{Re} z > 0$. We have

$$Q_5(z) = z^5\left(1 + \frac{1}{z} + \frac{2}{z^2} - \frac{8}{z^4} - \frac{1}{z^5}\right).$$

Whence,

$$\begin{aligned}\operatorname{Arg} Q_5(z) &= \operatorname{Arg}\left[z^5\left(1+\frac{1}{z}+\frac{2}{z^2}-\frac{8}{z^4}-\frac{1}{z^5}\right)\right]\\ &= \operatorname{Arg} z^5+\operatorname{Arg}\left(1+\frac{1}{z}+\frac{2}{z^2}-\frac{8}{z^4}-\frac{1}{z^5}\right)\\ &= 5\operatorname{Arg} z+\operatorname{Arg}\left(1+\frac{1}{z}+\frac{2}{z^2}-\frac{8}{z^4}-\frac{1}{z^5}\right).\end{aligned}$$

The variation of $\operatorname{Arg} Q_5(z)$ as z traverses the semicircle C_R in the positive sense is

$$\begin{aligned}\Delta_{C_R}\operatorname{Arg} Q_5(z) &= 5\Delta_{C_R}\operatorname{Arg} z\\ &\quad+\Delta_{C_R}\operatorname{Arg}\left(1+\frac{1}{z}+\frac{2}{z^2}-\frac{8}{z^4}-\frac{1}{z^5}\right).\end{aligned}$$

We now send R to ∞ and find that

$$\begin{aligned}&\lim_{R\to\infty}\Delta_{C_R}\operatorname{Arg} Q_5(z)\\ &=5\lim_{R\to\infty}\Delta_{C_R}\operatorname{Arg} z+\lim_{R\to\infty}\Delta_{C_R}\operatorname{Arg}\left(1+\frac{1}{z}+\frac{2}{z^2}-\frac{8}{z^4}-\frac{1}{z^5}\right).\end{aligned}$$

Both limits on the right-hand side exist and are, respectively,

$$\lim_{R\to\infty}\Delta_{C_R}\operatorname{Arg} z=\pi,$$

$$\lim_{R\to\infty}\Delta_{C_R}\operatorname{Arg}\left(1+\frac{1}{z}+\frac{2}{z^2}-\frac{8}{z^4}-\frac{1}{z^5}\right)=0.$$

Thus,

$$\lim_{R\to\infty}\Delta_{C_R}\operatorname{Arg} Q_5(z)=5\pi.$$

Now let point z move along the imaginary axis from $z = iR$ to $z = -iR$. We put $z = it$, $-R \leqslant t \leqslant R$. Then

$$Q_5(it) = u(t) + iv(t) = t^4 - 1 + i(t^5 - 2t^3 - 8t),$$

from which

$$\begin{cases} u = t^4 - 1, \\ v = t^5 - 2t^3 - 8t. \end{cases} \tag{11.1}$$

These are the parametric equations of a curve traversed by point $w = Q_5(z)$ in the (u, v) plane as point z traverses the imaginary axis downward. To build the curve we must find the points of its intersection with the u and v axes. Nulli-

fying u and v, we obtain, respectively,

$$t^4 - 1 = 0, \text{ or } t = \pm 1. \tag{11.2}$$

and

$$t^5 - 2t^3 - 8t = 0, \text{ or } t = \pm 2, \ t = 0. \tag{11.3}$$

Note that Eqs. (11.2) and (11.3) have no common roots that are real, i.e. the polynomial $Q_5(z)$ has no zeros on the imaginary axis. Consequently, we may apply the argument

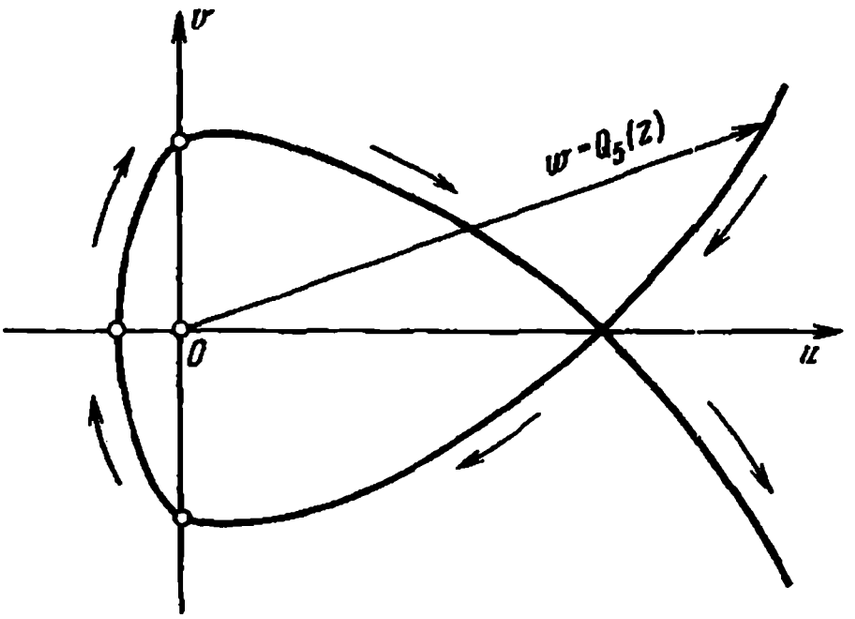


Figure 12

principle to the closed contour. We arrange the roots of Eqs. (11.2) and (11.3) in decreasing order, i.e. as the contour is circuited, and find the corresponding values of u and v. The result is given in the table below. Moreover,

$$\lim_{t \to \pm\infty} u = +\infty, \quad \lim_{t \to \pm\infty} v = \pm\infty.$$

No	t	u	v
1	2	15	0
2	1	0	−9
3	0	−1	0
4	−1	0	9
5	−2	15	0

With this data we can build the curve in the (u, v) plane (see Fig. 12). We see that the vector $w = Q_5(z)$ rotates through an angle $\varphi = 3\pi$ in the negative sense. Consequently,

$$\Delta_C \operatorname{Arg} Q_5(z) = 5\pi - 3\pi = 2\pi,$$

from which it follows that the number of zeros in the right half-plane is

$$N = \frac{2\pi}{2\pi} = 1.$$

Example 5. Find the number of zeros of the equation

$$Q_7(z) \equiv z^7 - 2z - 5 = 0$$

in the right half-plane.

Solution. We choose the same contour C as in Example 4. Then

$$\begin{aligned}
\Delta_{C_R} \operatorname{Arg} Q_7(z) &= \Delta_{C_R} \operatorname{Arg} (z^7 - 2z - 5) \\
&= \Delta_{C_R} \operatorname{Arg} \left[z^7 \left(1 - \frac{2}{z^6} - \frac{5}{z^7} \right) \right] \\
&= 7\Delta_{C_R} \operatorname{Arg} z + \Delta_{C_R} \operatorname{Arg} \left(1 - \frac{2}{z^6} - \frac{5}{z^7} \right) \\
&= 7\pi + \Delta_{C_R} \operatorname{Arg} \left(1 - \frac{2}{z^6} - \frac{5}{z^7} \right) \\
&\to 7\pi \text{ as } R \to \infty.
\end{aligned}$$

We put $z = it$, where $-R \leqslant t \leqslant R$. Then

$$Q_7(it) = u(t) + iv(t) = -5 + i(-t^7 - 2t),$$

whence

$$\begin{cases} u = -5, \\ v = -t(t^6 + 2). \end{cases}$$

Since $u \neq 0$, we can apply the argument principle ($Q_7(z)$ has no zeros on the imaginary axis). The curve in the (u, v) plane is a straight line (Fig. 13). The vector $w = Q_7(z)$ rotates through an angle π in the negative sense, which means that

$$\Delta_{C_R} \operatorname{Arg} Q_7(z) \xrightarrow[R \to \infty]{} 7\pi - \pi = 6\pi$$

and

$$N = \frac{6\pi}{2\pi} = 3,$$

i.e. the given equation has three roots in the right half-plane.

Determine the number of roots in the right half-plane of the following equations:

425. $z^4 + 2z^3 + 3z^2 + z + 2 = 0$.

426. $z^3 - 2z - 5 = 0$.

427. $z^3 - 4z^2 + 5 = 0$. 428. $2z^3 - z^2 - 7z + 5 = 0$.

429. $z^5 + 5z^4 - 5 = 0$. 430. $z^{12} - z - 1 = 0$.

Rouche's theorem *Suppose functions $f(z)$ and $\varphi(z)$ that are analytic in a closed domain D bounded by contour C are*

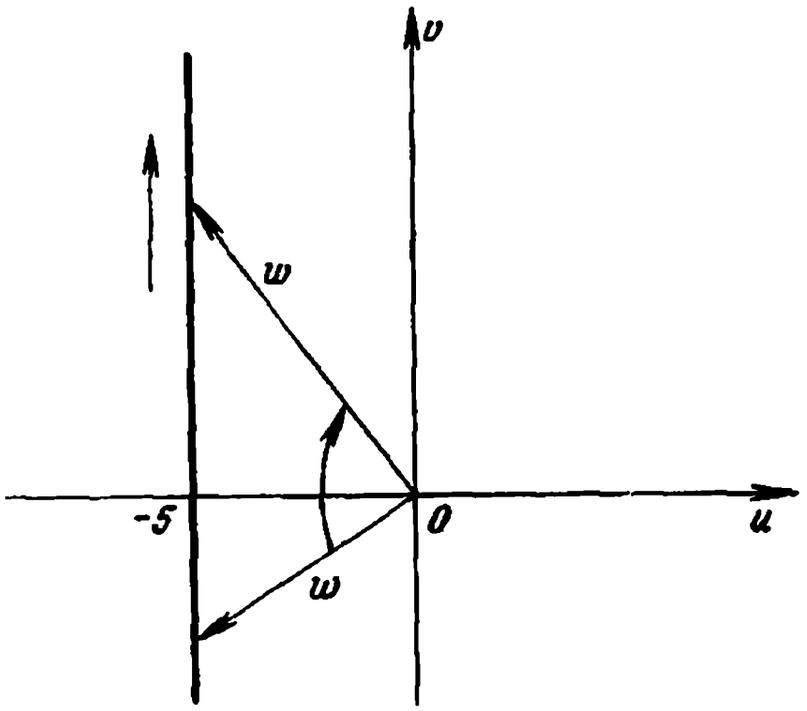


Figure 13

such that $|f(z)| > |\varphi(z)|$ at all points of C. Then their sum $F(z) = f(z) + \varphi(z)$ and the function $f(z)$ have the same number of zeros in D (with due regard for the order of the zeros).

Example 6. Find the number of zeros of the function

$$F(z) = z^8 - 4z^5 + z^2 - 1$$

inside the unit circle $|z| < 1$.

Solution. We write $F(z)$ as the sum of two functions, $f(z)$ and $\varphi(z)$, which we choose, for instance, in the following way:

$$f(z) = -4z^5, \quad \varphi(z) = z^8 + z^2 - 1.$$

Then on the circle $|z| = 1$ we have

$$|f(z)| = |-4z^5| = 4,$$

$$|\varphi(z)| = |z^8 + z^2 - 1| \leqslant |z^8| + |z^2| + 1 = 3.$$

Thus, on the boundary $|z| = 1$ we have $|f(z)| > |\varphi(z)|$. The function $f(z)$ has a fifth-order zero at the origin of coordinates. By virtue of Rouche's theorem, the function

$$F(z) = f(z) + \varphi(z) = z^8 - 4z^5 + z^2 - 1$$

has five zeros in the circle $|z| < 1$. Note that there is another way in which we can select the functions $f(z)$ and $\varphi(z)$, e.g.

$$f(z) = z^8 - 4z^5, \quad \varphi(z) = z^2 - 1.$$

Example 7. Find the number of roots of the equation

$$z^6 - 6z + 10 = 0$$

in the circle $|z| < 1$.

Solution. We take $f(z) = 10$ and $\varphi(z) = z^6 - 6z$. On the circle $|z| = 1$ we have

$$|f(z)| = 10, \quad |\varphi(z)| = |z^6 - 6z| \leqslant |z^6| + 6|z| = 7.$$

Thus at all points of the circle $|z| = 1$ we have $|f(z)| > |\varphi(z)|$. The function $f(z) = 10$ has no zeros in the circle $|z| < 1$, which means that by virtue of Rouche's theorem the function $z^6 - 6z + 10$ has no zeros in this circle either.

Employing Rouche's theorem, find the number of roots of the following equations in the stated domains:

431. $z^4 - 3z^3 - 1 = 0$, $|z| < 2$.

432. $z^3 + z + 1 = 0$, $|z| < \frac{1}{2}$.

433. $z^5 + z^2 + 1 = 0$, $|z| < 2$.

434. $z^8 + 6z + 10 = 0$, $|z| < 1$.

435. $27z^{11} - 18z + 10 = 0$, $|z| < 1$.

436. $z^8 - 6z^6 - z^3 + 2 = 0$, $|z| < 1$.

Example 8. How many roots of the equation

$$z^4 - 5z + 1 = 0 \tag{11.4}$$

are there in the annulus $1 < |z| < 2$?

Solution. Suppose N is the number of roots of Eq. (11.4) in the annulus $1 < |z| < 2$. Then $N = N_2 - N_1$, where N_1 is the number of roots of Eq. (11.4) in the circle $|z| < 1$ and N_2 is the number of roots of the same equation in the circle $|z| < 2$ ($N_2 \geqslant N_1$). We can easily see that

Eq. (11.4) has no roots on the circle $|z| = 1$, since $|z^4 - 5z + 1| \geqslant 3$ if $|z| = 1$. To find N_1 we take $f(z) = -5z$ and $\varphi(z) = z^4 + 1$. On the circle $|z| = 1$ we have $|f(z)| > |\varphi(z)|$, since $|f(z)| = |-5z| = 5$ and $|\varphi(z)| = |z^4 + 1| \leqslant |z^4| + 1 = 2$. The function $f(z) = -5z$ has one zero in the circle $|z| < 1$, which means that $N_1 = 1$. To find N_2 we take $f(z) = z^4$ and $\varphi(z) = 1 - 5z$. On the circle $|z| = 2$ we have $|f(z)| > |\varphi(z)|$, since $|f(z)| = |z^4| = 2^4 = 16$ and $|\varphi(z)| = |1 - 5z| \leqslant 1 + 5|z| = 11$. The function $f(z) = z^4$ has four roots in the circle $|z| < 2$, which means that $N_2 = 4$. Thus, the number of roots of Eq. (11.4) in the annulus $1 < |z| < 2$ is $N = 4 - 1 = 3$.

Determine the number of roots for each of the equation given below in the stated annuluses:

437. $4z^4 - 29z^2 + 25 = 0$, $2 < |z| < 3$.
438. $z^7 - 5z^4 + z^2 - 2 = 0$, $1 < |z| < 2$.
439. $z^6 - 8z + 10 = 0$, $1 < |z| < 3$.

Example 9. Find the number of roots of the equation

$$z^2 - ae^z = 0, \text{ where } 0 < a < e^{-1},$$

in the unit circle $|z| < 1$.

Solution. We put $f(z) = z^2$ and $\varphi(z) = -ae^z$. On the circle $|z| = 1$ we have

$$|f(z)| = |z^2| = 1,$$

$$|\varphi(z)| = |-ae^z| = a|e^z| = a|e^{x+iy}| = ae^x \leqslant ae < 1,$$

by virtue of the conditions $-1 \leqslant x \leqslant 1$ and $0 < a < e^{-1}$. Thus, $|f(z)| > |\varphi(z)|$ if $|z| = 1$. The function $f(z) = z^2$ has a second-order root at the origin of coordinates in the circle $|z| < 1$, which means that by virtue of Rouche's theorem the initial equation has two roots in the same circle.

Remark. Let us take a real-valued function $F(x) = x^2 - ae^x$. It is continuous on the segment $-1 \leqslant x \leqslant 1$.

Moreover,

$$F(-1) = 1 - ae^{-1} > 0, \text{ since } 0 < ae^{-1} < e^{-2} < 1,$$

$$F(0) = -a < 0,$$

$$F(1) = 1 - ae > 0, \text{ since } a < e^{-1}.$$

Thus, at the ends of the segments $-1 \leqslant x \leqslant 0$ and $0 \leqslant x \leqslant 1$ the function $F(x)$ assumes values with different signs. This means that the given equation has two real roots of opposite signs in the circle $|z| < 1$.

Determine the number of roots of the following equations in the given domains:

440. $e^{z-\lambda} = z$ $(\lambda > 1)$, $|z| < 1$.

441. $e^z = az^n$, where n is a positive integer, $|a| > e^R/R^n$, and $|z| < R$.

442. $z^2 - \cos z = 0$, $|z| < 2$.

443. $z^4 - \sin z = 0$, $|z| < \pi$.

444. $z^2 + \cosh iz = 0$, $|z| < 0.5$.

445. $\cosh z = z^2 - 4z$, $|z| < 1$.

446. $2^z = 4z$, $|z| < 1$.

Example 10. Find the number of roots of the equation

$$\lambda - z - e^{-z} = 0, \quad \lambda > 1,$$

in the right half-plane $\operatorname{Re} z > 0$.

Solution. We take a contour consisting of the segment $[-iR, iR]$ and the right semicircle $|z| = R$. We put $f(z) = z - \lambda$ and $\varphi(z) = e^{-z}$. On the segment $[-iR, iR]$, where $z = iy$, we have

$$|f(z)| = |iy - \lambda| = \sqrt{\lambda^2 + y^2} \geqslant \sqrt{\lambda^2} = \lambda > 1,$$

$$|\varphi(z)| = |e^{-z}| = |e^{-iy}| = 1,$$

and hence $|f(z)| > |\varphi(z)|$. On the semicircle $|z| = R$ with $\operatorname{Re} z = x > 0$ we have $|f(z)| > |\varphi(z)|$, provided R is large $(R > \lambda + 1)$, since

$$|f(z)| = |z - \lambda| \geqslant |z| - \lambda = R - \lambda > 1,$$

$$|\varphi(z)| = |e^{-z}| = |e^{-x-iy}| = |e^{-x}e^{-iy}| = e^{-x}\,|e^{-iy}|$$

$$= e^{-x} \leqslant 1 \quad (x > 0).$$

By Rouche's theorem, inside the given contour the equation has as many roots with an arbitrarily large R as the equation $f(z) = z - \lambda = 0$, i.e. one root. That is in the entire right half-plane the given equation has only one root.

447. Show that the equation $ze^{\lambda - z} = 1$, with $\lambda > 1$, has only one real, positive root on the unit circle $|z| \leqslant 1$.

448. Show that the equation $1 + z + \alpha z^n = 0$, with n a positive integer greater than unity, for any α has at least one root in the circle $|z| \leqslant 2$.

449. Let $f(z)$ and $\varphi(z)$ be analytic in a neighborhood of a point a and let C be a circle centered at a and such that

$$|\alpha f(z)| + |\beta\varphi(z)| < r$$

on the circumference of this circle. Show that the equation $F(z) = z - a - \alpha f(z) - \beta f(z) = 0$ has one and only one root in C.

§ 12 Conformal Mapping

The idea of a conformal mapping

Definition. A mapping of the neighborhood of a point z_0 onto the neighborhood of a point w_0 performed by a function $w = f(z)$ is said to be *conformal* if at point z_0 the mapping

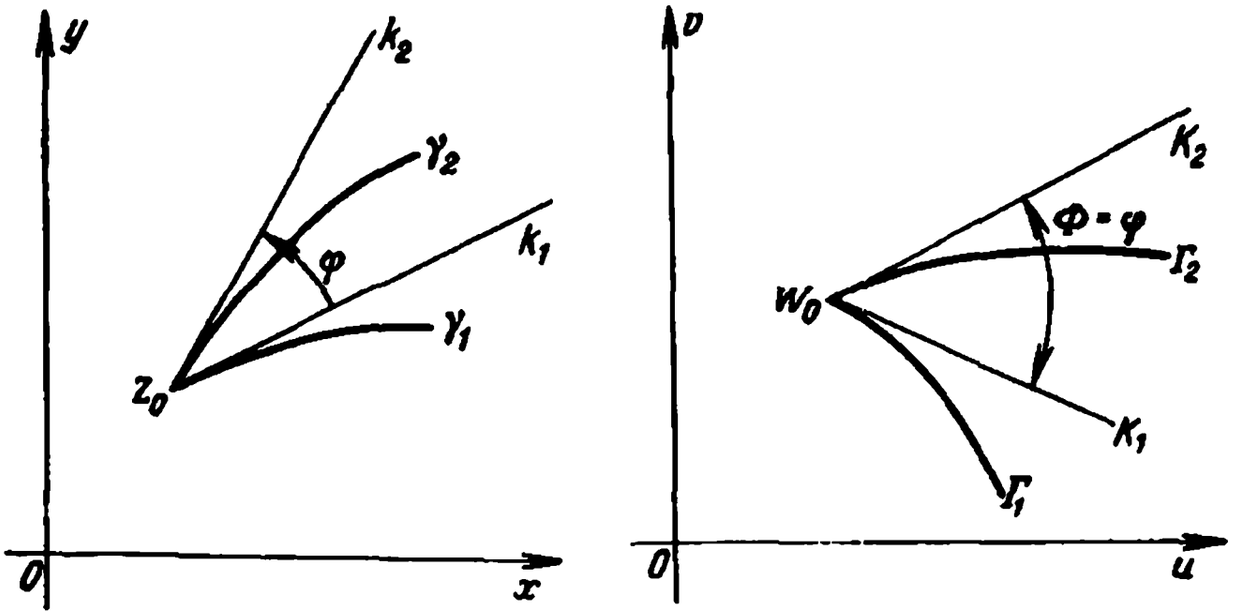


Figure 14

preserves the angles between lines and retains constancy of stretching (Fig. 14). This means that (1) if under the mapping performed by $w = f(z)$ the curves γ_1 and γ_2 are mapped into curves Γ_1 and Γ_2, then the angle φ between the tangents k_1 and k_2 to the curves γ_1 and γ_2 at point z_0 is equal to the angle Φ between the corresponding tangents K_1 and K_2 to Γ_1 and Γ_2 at point w_0, i.e. $\Phi = \varphi$, and (2) if we take an infinitely small circle centered at z_0 in the complex z plane, then in the complex w plane the result of the mapping is an infinitely small circle centered at w_0. For this reason we say that a *conformal mapping preserves angles and retains similarity in the small.*

If not only the magnitudes of the angles but the sense of rotation remains the same under a mapping performed by a function $w = f(z)$, we speak of *conformal mapping of the first kind*; but if the sense of rotation changes to opposite, we speak of *conformal mapping of the second kind*. A simple example of a conformal mapping of the first kind is that performed by the function $w = z$, while $w = \bar{z}$ is an example of a conformal mapping of the second kind. From now on we will deal only with mappings of the first kind.

If a mapping is conformal at all points of a domain D, we speak of it as being *conformal in* D.

The conformality criterion *A mapping performed by a function* $w = f(z)$ *is conformal in a domain* D *if and only if this function is univalent and analytic in* D, *with* $f'(z) \neq 0$ *for every* $z \in D$. (A function $w = f(z)$ is said to be *univalent*, or *simple*, *in* D if at different points in D it admits different values.) If the univalence of $f(z)$ is not presupposed, the mapping performed by this function will not necessarily be one-to-one and thus may not be conformal. For instance, the function $w = z^4$ is analytic everywhere in the semiannulus $1 \leqslant |z| \leqslant 2, 0 \leqslant \arg z \leqslant \pi$ and in addition $w' = 4z^3 \neq 0$ in the same domain, but it maps the given semiannulus onto $1 \leqslant w \leqslant 16, 0 \leqslant \arg w \leqslant 4\pi$, which is a domain that covers the corresponding annulus in the complex w plane twice, i.e. the requirement that the mapping be one-to-one is violated.

Example 1. In what domains D are the mappings performed by the functions

$$\text{(a) } w = 2z, \text{ (b) } w = (z-2)^2$$

conformal?

Solution. (a) Since $f(z) = 2z$ is analytic and univalent in the entire complex z plane and its derivative $f'(z) = 2 \neq 0$, the mapping is conformal in the entire complex plane.

(b) The mapping $w = (z-2)^2$ is conformal everywhere except at point $z = 2$, where $f'(z) = 2(z-2)$ vanishes.

450. Determine the domains of conformality of the following mappings:

(a) $w = e^{-3z}$; (b) $w = z^2 - 4z$; (c) $w = -iz^2$;

(d) $w = \sinh(1 - z)$; (e) $w = (z + 2i)^3$.

General theorems of conformal mapping theory

1. Riemann's theorem *There is always an analytic function $w = f(z)$ that maps conformally and in a one-to-one manner a simply connected, smooth domain D into another domain G provided that neither of the two domains coincides with the entire complex plane without one point or with the entire extended plane (i.e. the plane with the point at infinity).*

Generally, there is an infinitude of analytic functions that map a given domain D onto a given domain G, but if we wish that only one such function exist, one of the following conditions must be met:

(a) a given point z_0 of D is mapped into a given point w_0 of G and a curve emerging from z_0 is rotated through a given angle α (here $w_0 = f(z_0)$ and $\alpha = \arg f'(z_0)$);

(b) a point z_0 of D and a point z_1 of the boundary γ of D are mapped, respectively, into a point w_0 of G and a point w_1 of the boundary Γ of G (here $w_0 = f(z_0)$ and $w_1 = f(z_1)$);

(c) three points z_1, z_2, and z_3 of the boundary γ of D are mapped, respectively, into three points w_1, w_2, and w_3 of the boundary Γ of G (here $w_1 = f(z_1)$, $w_2 = f(z_2)$, and $w_3 = f(z_3)$) and in such a way that if as point z is moving along γ from z_1 to z_2 to z_3 the domain D is to the left (right), then as point w is moving along Γ from w_1 to w_2 to w_3 the domain G is also to the left (right).

In cases (b) and (c) the function $f(z)$ is assumed to be continuous in the closed domain D.

2. *The principle of one-to-one correspondence between boundaries.* Suppose a domain D is bounded by a smooth or piecewise smooth contour γ. Let a function $w = f(z)$ that is analytic in D and on γ map γ onto a contour Γ that is the boundary of a domain G. Suppose, finally, that as point z traverses γ in such a way that D is to the left the corresponding point w traverses Γ in such a way that G is to the left. Then function $w = f(z)$ maps D conformally and in a one-to-one manner onto G.

3. *The symmetry principle.* Suppose a domain D whose boundary contains a straight section γ (finite or infinite) is mapped by a function $w = f(z)$ onto a domain G in such a way that γ is mapped into a straight segment Γ that is part of the boundary of G (Fig. 15). We denote the straight

lines that contain γ and Γ by l and L, respectively. The symmetry principle states that if $w = f(z)$ is analytic in D and at all interior points of the segment γ, then it is analytic in the domain D^* that is symmetric to D with respect to l and maps any two points z_1 and z_2 (one of which lies in

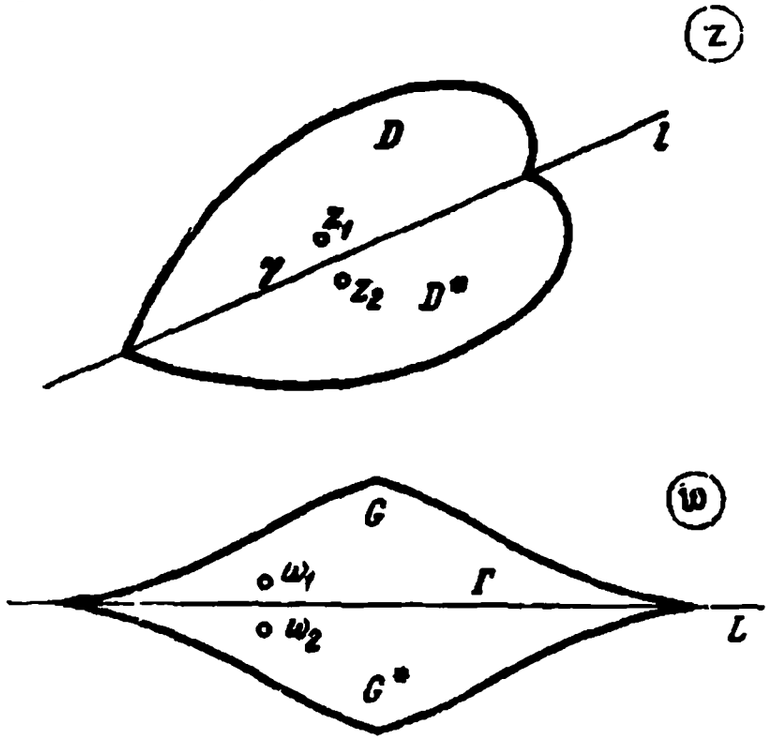


Figure 15

D) that are symmetric with respect to l into points w_1 and w_2 symmetric with respect to L.

Example 2. Suppose the function

$$w = 3z + i$$

is defined in a domain D bounded by the contour γ:

$$x^2 + y^2 - 2x = 0.$$

Find the domain onto which the above function maps D.

Solution. Let $z = x + iy$ and $w = u + iv$. Then we can write $u + iv = 3x + i(3y + 1)$, so that $u = 3x$ and $v = 3y + 1$. This means that $x = u/3$ and $y = (v - 1)/3$. Contour γ is mapped into Γ:

$$\left(\frac{u}{3}\right)^2 + \left(\frac{v-1}{3}\right)^2 - 2\frac{u}{3} = 0, \text{ or } (u-3)^2 + (v-1)^2 = 9,$$

which is a circle of radius 3 centered at point $M(3, 1)$. The positive sense of traversal of γ corresponds to the positive

sense of traversal of Γ. This can be verified by writing the equations of the respective contours in parametric form:

$$\gamma : x = 1 + \cos\varphi, \quad y = \sin\varphi, \qquad 0 \leqslant \varphi < 2\pi,$$

$$\Gamma : u = 3 + 3\cos\varphi, \quad v = 3\sin\varphi + 1, \quad 0 \leqslant \varphi < 2\pi.$$

According to the principle of one-to-one correspondence between boundaries, D is mapped into G, which is the interior of the circle bounded by Γ. Another way to verify this is to proceed as follows. Take a point $z \in D$ and its image under the map $w = 3z + i$. For instance, point $z = 1$ becomes point $w = 3 + i$, which lies inside Γ.

Example 3. Given two points $z_1 = 2 + 3i$ and $z_2 = 3 + 2i$ symmetric with respect to the straight line $y = x$. Show that the function $w = e^{-i\pi/2}z$ maps points z_1 and z_2 into points $w_1 = 3 - 2i$ and $w_2 = 2 - 3i$ symmetric with respect to the straight line $y = -x$.

Solution. We can easily see that $w = e^{-i\pi/2}z$ maps the straight line $y = x$ into the straight line $y = -x$ and is analytic in the entire complex plane. By virtue of the symmetry principle, the points $z_1 = 3 + 2i$ and $z_2 = 2 + 3i$, which are symmetric with respect to the straight line $y = x$, are mapped by this function into points $w_1 = 3 - 2i$ and $w_2 = 2 - 3i$, which are symmetric with respect to the straight line $y = -x$.

Example 4. Show that the function $w = e^{\pi z/h}$ maps the strip $0 < \operatorname{Im} z < h$ onto the upper half-plane $\operatorname{Im} w > 0$.

Solution. We will traverse the boundary of D in such a way that D is to the left. Since

$$w = u + iv = e^{\frac{\pi(x+iy)}{h}} = e^{\frac{\pi x}{h}} e^{\frac{i\pi y}{h}},$$

we find that when point z traverses the real axis from $x = -\infty$ to $x = +\infty$ (with $y = 0$), the corresponding point $w = e^{\pi x/h}$ traverses the real positive semiaxis in the complex w plane from point $u = 0$ to point $u = +\infty$ (with $v = 0$). When point $z = x + ih$ traverses the upper boundary of the strip from point $+\infty + ih$ to point $-\infty + ih$ (with x varying from $+\infty$ to $-\infty$), the corresponding point $w = e^{\pi x/h}e^{i\pi} = -e^{\pi x/h}$ traverses the real negative semiaxis in the complex w plane from point $-\infty$ to point 0. Since $w = e^{\pi z/h}$ is analytic in D: $0 < \operatorname{Im} z < h$ and on

the boundary of D, it maps this domain conformally onto G: $\operatorname{Im} w > 0$.

451. Show that the semiannulus $1 \leqslant |z| \leqslant 2$, $0 \leqslant \arg z \leqslant \pi$ is mapped by the function $w = z^2$ onto the annulus $1 \leqslant |w| \leqslant 4$, $0 \leqslant \arg w \leqslant 2\pi$.

452. Show that the angle $0 < \arg z < \pi/5$, $0 < |z| < +\infty$ is mapped by the function $w = z^5$ onto the upper half-plane $\operatorname{Im} w > 0$, so that point $z = 0$ is mapped into point $w = 0$.

453. Show that the strip $1 \leqslant y < 1 + 2\pi$ is mapped by the function $w = e^z$ onto the entire complex w plane with a cut along the real positive semiaxis.

Conformal mapping performed by the linear function $w = az + b$, the function $w = 1/z$, and the linear-fractional function $w = \dfrac{az+b}{cz+d}$

1. The linear function. This is the function $w = az + b$, with a and b constant complex numbers ($a \neq 0$). It performs conformal mapping of the entire complex z plane onto the entire complex w plane, since for each z we have $w' = a \neq 0$. Particular cases are

$$w = z + b, \tag{12.1}$$

which performs translations, the function

$$w = e^{i\alpha} z \tag{12.2}$$

(with α real), which performs rotations about the origin of coordinates through the angle α, and the function

$$w = rz \tag{12.3}$$

(with r real and positive), which performs similarity transformations with the center of similitude at the origin of coordinates and r the ratio of similitude. The general linear mapping

$$w = az + b, \text{ where } a = re^{i\alpha}, \tag{12.4}$$

can be broken down into successive (a) rotation about the origin of coordinates through the angle α, (b) similarity transformation with the center of similitude at the origin of coordinates and with a ratio of similitude equal to r, and

(c) translation by the vector corresponding to the complex number b.

Note that a linear transformation leaves two points fixed, $z_1 = \infty$ and $z_2 = b/(1 - a)$. For $a = 1$, $z_2 = \infty$, and the two points merge.

Example 5. Show that the linear transformation $w = az + b$ is completely defined if two different points z_1 and z_2 are mapped, under this transformation, into arbitrarily fixed but different points w_1 and w_2, respectively.

Solution. Indeed, the mapping $w = az + b$ is defined if we know the parameters a and b. We wish to show that the hypothesis enables finding these parameters in a unique manner. Suppose for $z = z_1$ we obtain $w = w_1$, i.e. $w_1 = az_1 + b$, and for $z = z_2$ we have $w_2 = az_2 + b$. This yields

$$a = \frac{w_2 - w_1}{z_2 - z_1}, \quad b = \frac{w_1 z_2 - w_2 z_1}{z_2 - z_1} \quad (z_1 \neq z_2).$$

These relations completely define the parameters a and b.

Example 6. Show that the linear transformation (12.4) is completely defined if point z_1 is mapped into point w_1 and the derivative dw/dz has the value a at z_1.

Solution. To completely define mapping (12.4), we must fix the parameters a and b. From the condition that z_1 is mapped into w_1 we find that $w_1 = az_1 + b$. Subtracting this from (12.4), we obtain $w - w_1 = a(z - z_1)$. Obviously $dw/dz = a$ for every z. For this reason, by fixing the value of dw/dz at point z_1 we define parameter a. Thus, the linear mapping $w - w_1 = a(z - z_1)$ is completely defined (b is simply $w_1 - az_1$).

Example 7. Find the linear function that maps the triangle with vertices at points 0, 1, and i in the complex z plane into a similar triangle with vertices at $1 + i$, 0, and 2 in the complex w plane.

Solution. Figure 16 shows that $\triangle ABC$ is mapped into a similar triangle $\triangle A_1B_1C_1$ via the following transformations:

(a) a rotation about the origin of coordinates through the angle $5\pi/4$, which corresponds to the transformation

$$w_1 = e^{i\frac{5}{4}\pi} z;$$

(b) a similarity transformation with the center of similitude at the origin of coordinates and the ratio of simili-

tude $r = \sqrt{2}$ (since $A_1B_1/AB = \sqrt{2}$):

$$w_2 = \sqrt{2}w_1;$$

and (c) a translation that shifts point C (0, 0) to point C_1 (1, 1) (we obtain $b = 1 + i$):

$$w = w_2 + 1 + i.$$

Taking into account that $e^{i\pi 5/4} = -\sqrt{2}/2 - i\sqrt{2}/2$, we

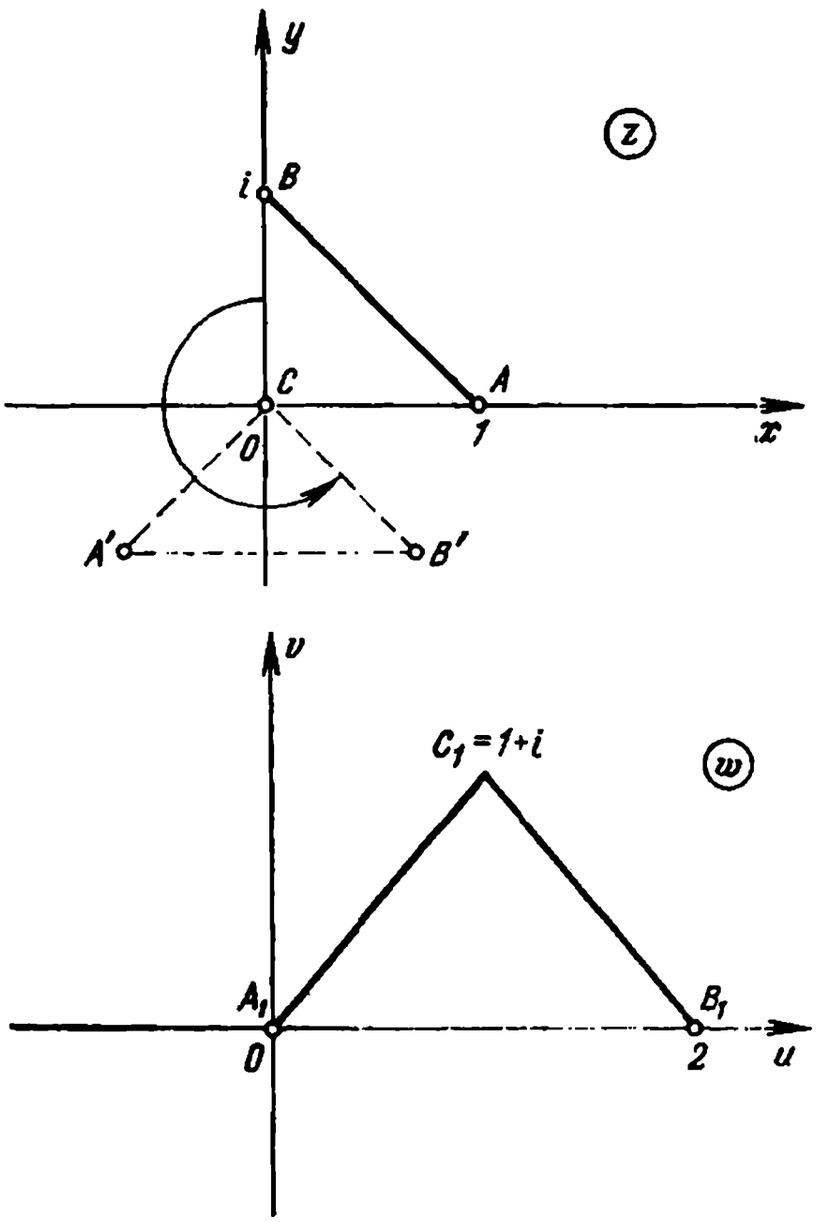


Figure 16

finally obtain

$$w = \sqrt{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)z + 1 + i = (1 - z)(1 + i).$$

Alternative solution. Suppose the sought function is $w = az + b$, where a and b are constants yet to be found. By the hypothesis, the function must map the points $z_1 = 0$

and $z_2 = 1$ into the points $w_1 = 1 + i$ and $w_2 = 0$. This yields the following system of equations for finding a and b:

$$\begin{cases} 1 + i = b, \\ 0 = a + b. \end{cases}$$

Whence, $a = -1 - i$ and $b = 1 + i$, which means that

$$w = (1 + i)(1 - z).$$

454. Establish the geometric meaning (translation, similarity transformation, rotation) of the following trans-

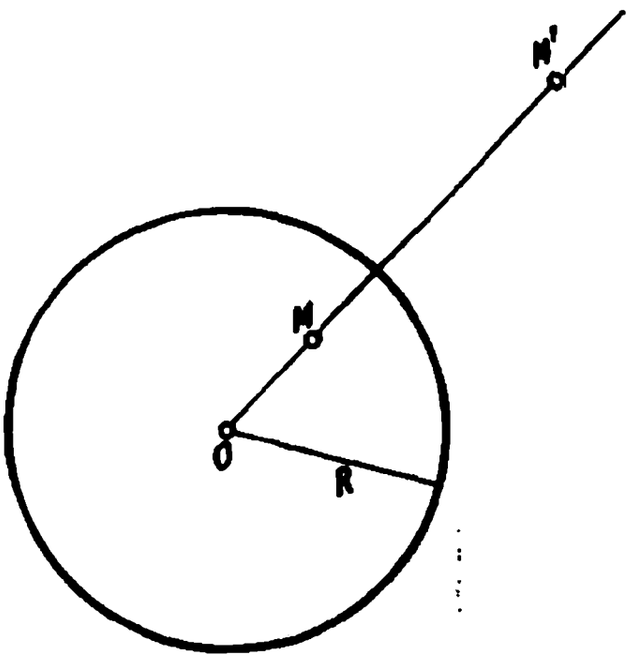


Figure 17

formations:

(a) $w = z + 3i$; (b) $w = z + 5$; (c) $w = iz$; (d) $w = e^{i\pi/6}z$;
(e) $w = 3z$; (f) $w = (1 - i)\, z/\sqrt{2}$.

455. Find the general form of the linear functions that perform the following transformations:

(a) the upper half-plane onto itself;
(b) the upper half-plane onto the lower half-plane;
(c) the upper half-plane onto the right half-plane.

456. Find the linear mapping $w = az + b$ that leaves the point z_0 stationary and maps point z_1 into point w_1:

(a) $z_0 = 1 - i$, $z_1 = 2 + i$, $w_1 = 4 - 3i$;
(b) $z_0 = -i$, $z_1 = 1 - 2i$, $w_1 = 2 - 3i$;
(c) $z_0 = -1 - i$, $z_1 = 3 - 2i$, $w_1 = 3i$.

457. Find the linear function $w = f(z)$ that maps the strip between the straight lines $x = a$ and $x = a + h$ onto the strip $0 < u < 1$ in the complex w plane.

2. The function

$$w = \frac{1}{z}. \tag{12.5}$$

Points M and M' are said to be *symmetric with respect to circle (circumference)* Γ if

(a) both lie on the same ray that emerges at the center of circle;

and (b) the product of their distances from the center of the circle is equal to the square of the circle's radius: $OM \times \times OM' = R^2$ (Fig. 17).

Remark. Points on the circumference Γ are symmetric to themselves with respect to Γ.

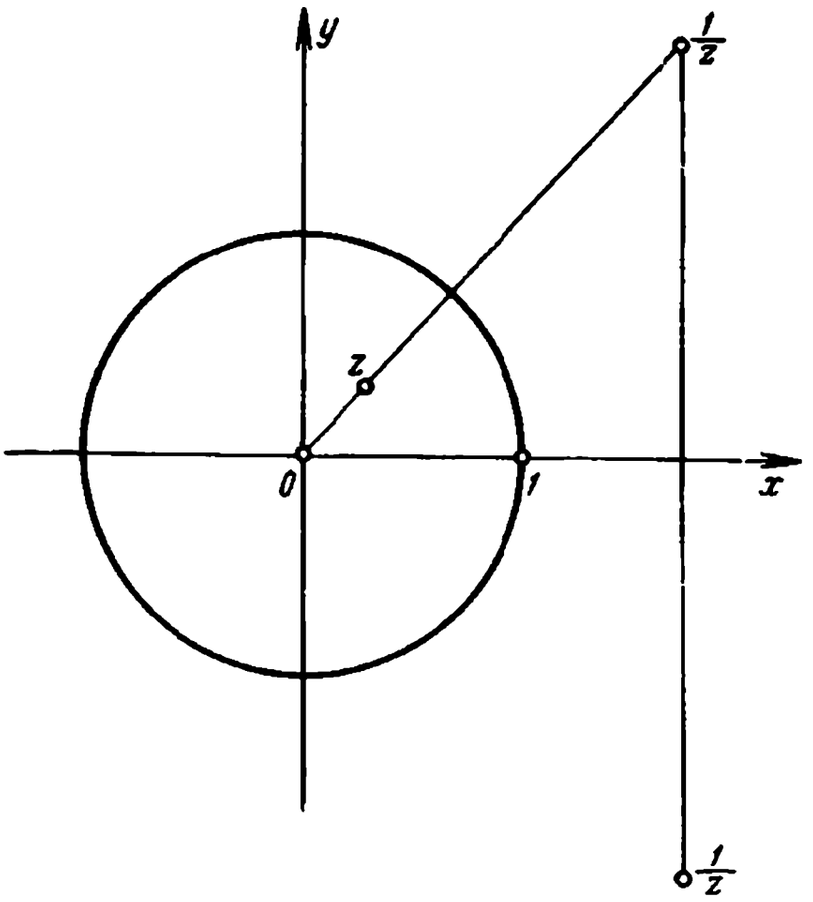


Figure 18

The point that is symmetric to the center O of Γ with respect to Γ is the point at infinity.

If the center of Γ lies at the origin of coordinates and one of the symmetric points with respect to Γ corresponds to the

complex number z, the other corresponds to the complex number $R^2/\bar{z}$.

The mapping $w = 1/z$ consists of two symmetric reflections: one with respect to the unit circle and the other with respect to the real axis (Fig. 18); we speak of this mapping as an *inversion*. It is conformal in the entire extended plane, with point $z = 0$ corresponding to point $w = \infty$ and point $z = \infty$ to point $w = 0$. (We assume that the angle between curves at the point at infinity in one of the planes (z or w) is equal to the angle between the images of these curves at the origin of coordinates in the other plane.) Circles (or straight lines) are mapped by $w = 1/z$ into circles (or straight lines). The stationary (fixed) points are $z_1 = +1$ and $z_2 = -1$.

Example 8. Find the map of the circle $|z| = 3$ as a result of applying the mapping $w = 25/z$.

First solution. Let $z = x + iy$ and $w = u + iv$. Then we can write $w = 25/z$ in the following form:

$$u + iv = \frac{25}{x+iy} = \frac{25x}{x^2+y^2} - i\,\frac{25y}{x^2+y^2},$$

whence

$$u = \frac{25x}{x^2+y^2}, \quad v = -\frac{25y}{x^2+y^2}. \tag{a}$$

In Cartesian coordinates the equation of the circle $|z| = 3$ is

$$x^2 + y^2 = 9. \tag{b}$$

Excluding x and y from (a) and (b), we find that

$$u^2 + v^2 = \left(\frac{25}{3}\right)^2,$$

which is a circle of radius $R = 25/3$ centered at the origin of coordinates on the complex w plane.

Second solution. We write z and w in exponential form:

$$z = \rho e^{i\varphi}, \quad w = re^{i\theta}.$$

Then under the mapping $w = 25/z$ we obtain

$$re^{i\theta} = \frac{25}{\rho e^{i\varphi}},$$

whence $r = 25/\rho$ and $\theta = -\varphi$, where $\rho = 3$ and $0 \leqslant \varphi < 2\pi$. This yields

$$w = \frac{25}{3} e^{-i\varphi}$$

which is a circle of radius $r = 25/3$ centered at the origin of coordinates and traversed clockwise when the initial circle is traversed counterclockwise.

Third solution. The fact that $w = 25/z$ yields $z = 25/w$. Substituting this expression for z into the equation of the circle $|z| = 3$ and employing the property of the modulus of a complex quantity, we obtain

$$\left|\frac{25}{w}\right| = 3, \text{ or } \frac{25}{|w|} = 3,$$

whence $|w| = 25/3$. Consequently, the map of the circle $|z| = 3$ obtained as a result of applying $w = 25/z$ is the circle $|w| = 25/3$.

458. Find the domain into which the function $w = 1/z$ maps the half-strip $0 < \operatorname{Re} z < 1$, $\operatorname{Im} z > 0$.

459. Find the mass of the following sets obtained as a result of applying $w = 1/z$:

(a) $\arg z = \pi/3$; (b) $|z| = 1$, $\frac{\pi}{4} < \arg z < \pi$;

(c) $2 \leqslant x \leqslant 4$, $y = 0$; (d) $-2 < y < -1$, $x = 0$;

(e) $0 < \operatorname{Re} z < 1$.

3. The linear-fractional function

$$w = \frac{az + b}{cz + d}, \tag{12.6}$$

where a, b, c, and d are complex constants and $ad - bc \neq 0$, maps conformally and in a one-to-one manner the extended complex z plane onto the extended complex w plane. The mapping induced by the linear-fractional function is said to be *linear-fractional*. A linear-fractional mapping may be achieved by successively applying linear mappings and the mapping of the $w = 1/z$ type.

Example 9. Find the conditions in which the linear-fractional function (12.6) maps the upper half-plane $\operatorname{Im} z > 0$ onto the upper half-plane $\operatorname{Im} w > 0$.

Solution. The mapping requires that the boundary of the domain $\operatorname{Im} z > 0$, which is the real x axis traversed from

left to right, is mapped into the boundary of the domain Im $w > 0$, which is the real u axis traversed from left to right, too. Thus, for all real values of z the values of w must also be real. This, obviously, is possible only for real a, b, c, and d. Moreover, each $z = x + iy$, with $y > 0$, must have corresponding to it a $w = u + iv$ such that $v > 0$. Substituting $z = x + iy$ into (12.6), we obtain

$$w = u + iv = \frac{(ax+b)(cx+d)+acy^2}{(cx+d)^2+y^2} + i\frac{(ad-bc)y}{(cx+d)^2+y^2},$$

whence

$$v = \frac{(ad-bc)y}{(cx+d)^2+y^2}.$$

Since here $y > 0$ and the denominator is positive, the requirement that v be positive too is fulfilled if and only if $ad - bc > 0$. This is the sought condition.

Properties of the linear-fractional mapping

1. The circular property. The linear-fractional mapping maps a circle into a circle. (A straight line is assumed to be a circle of infinite radius.)

2. The symmetry property. Two points z_1 and z_2 that are symmetric with respect to a circle C are mapped by the linear-fractional mapping into points w_1 and w_2 symmetric with respect to the circle Γ into which C is mapped.

Corollary *If under the linear-fractional mapping* $w = f(z)$ *a straight line or circle* γ *is mapped into a circle* Γ *and one of two points symmetric with respect to* γ *is mapped into the center of* Γ, *the other is necessarily mapped into the point at infinity.*

3. There is only one linear-fractional function that maps three given points z_1, z_2, and z_3 in the complex z plane into three given points w_1, w_2, and w_3 in the complex w plane. The function is

$$\frac{w-w_1}{w-w_2}\frac{w_3-w_2}{w_3-w_1} = \frac{z-z_1}{z-z_2}\frac{z_3-z_2}{z_3-z_1}. \qquad (12.7)$$

Example 10. Find the linear-fractional function that maps points $z_1 = 1$, $z_2 = i$, and $z_3 = -1$ into points $w_1 = -1$, $w_2 = 0$, $w_3 = 1$.

Solution. Employing (12.7) we have

$$\frac{w+1}{w-0}\,\frac{1-0}{1-(-1)}=\frac{z-1}{z-i}\,\frac{-1-i}{-1-1},$$

whence

$$w=i\,\frac{i-z}{i+z}.$$

Remark. If one of the points z_k or w_k ($k=1, 2, 3$) is the point at infinity, then in (12.7) we must substitute unities for all differences containing this point.

Example 11. Find the linear-fractional function that maps point z_1 into $w_1=0$ and point z_2 into point $w_2=\infty$.

Solution. We take an arbitrary point z_3 different from z_1 and z_2 and assume that the function we are seeking maps z_3 into a point w_3 that is neither w_1 nor w_2. Then, bearing in mind (12.7) and the above remark, we obtain

$$\frac{w-0}{1}\,\frac{1}{w_3-0}=\frac{z-z_1}{z-z_2}\,\frac{z_3-z_2}{z_3-z_1},$$

whence

$$w=K\,\frac{z-z_1}{z-z_2},$$

where

$$K=\frac{z_3-z_2}{z_3-z_1}\,w_3, \tag{12.8}$$

i.e. K is an arbitrary complex number, $K\neq 0$.

Example 12. Map the upper half-plane $\operatorname{Im} z>0$ onto the unit circle $|w|<1$ in such a way that point z_0 (with $\operatorname{Im} z_0>0$) is mapped into the center $w=0$ of the circle.

Solution. Since point z_0 is mapped by the sought linear-fractional function $w=w(z)$ into the center of the circle, i.e. $w(z_0)=0$, the complex conjugate point $\bar{z}_0$ must be mapped into $w=\infty$ (by the symmetry property). Next we use (12.8) and obtain

$$w=K\,\frac{z-z_0}{z-\bar{z}_0},$$

where K is a constant factor. For all K's the function maps the upper half-plane onto a circle with its center at point $w=0$. Let us select K so that the circle is the unit circle.

It suffices to require that point $z = 0$ (the boundary point of the domain $\operatorname{Im} z > 0$) be mapped into a point on the unit circle $|w| = 1$. Then

$$1 = |w| = |K| \times \left|\frac{z_0}{\bar{z}_0}\right|, \text{ whence } |K| = 1, \text{ so that } K = e^{i\alpha},$$

where α is any real number. Thus,

$$w = e^{i\alpha} \frac{z - z_0}{z - \bar{z}_0}. \tag{12.9}$$

Remark. Let us find the derivative w' at point $z_0 = a + ib$ (with b positive):

$$w'(z_0) = -\frac{ie^{i\alpha}}{2b}, \text{ or } w'(z_0) = \frac{1}{2b} e^{i\alpha} e^{-i\pi/2} = \frac{e^{i(\alpha - \pi/2)}}{2b}.$$

This means that $\arg w'(z_0) = \alpha - \pi/2$, so that by the geometric meaning of a derivative the angle of rotation of curves at point z_0 under the mapping (12.9) is $\alpha - \pi/2$.

By Riemann's theorem, there is a unique mapping $w = w(z)$ of the upper half-plane $\operatorname{Im} z > 0$ onto the circle $|w| < 1$ that satisfies the following requirements: $w(z_0) = 0$ and $\arg w'(z_0) = \alpha - \pi/2$. This implies that each linear-fractional mapping of the half-plane $\operatorname{Im} z > 0$ onto the circle $|w| < 1$ has the form (12.9).

Example 13. Map the unit circle $|z| < 1$ onto the unit circle $|w| < 1$.

Solution. Let the sought linear-fractional function $w = w(z)$ map point z_0 in the circle $|z| < 1$ into the center of the circle $|w| < 1$, so that $w(z_0) = 0$. Then point $z_0^* = 1/\bar{z}_0$, which is symmetric to z_0 with respect to the unit circle $|z| = 1$, is mapped into point ∞, i.e. $w(z_0^*) = \infty$. Then (12.8) yields

$$w = K \frac{z - z_0}{z - 1/\bar{z}_0}, \text{ or } w = K_1 \frac{z - z_0}{1 - z\bar{z}_0},$$

where $K_1 = -K\bar{z}_0$ is a constant complex number. We select K_1 in such a way that the circle in the complex w plane is a unit circle. To this end it suffices to require that point $z = 1$ be mapped into a point on the unit circle $|w| = 1$. We then have

$$1 = |w| = |K_1| \left|\frac{1 - z_0}{1 - \bar{z}_0}\right|,$$

whence $|K_1| = 1$ since $|1 - z_0| = |1 - \bar{z}_0|$. Consequently, $K_1 = e^{i\alpha}$, where α is arbitrary but real. Thus,

$$w = e^{i\alpha} \frac{z - z_0}{1 - z\bar{z}_0}, \tag{12.10}$$

where $|z_0| < 1$, with α arbitrary but real.

Remark. Since $w'(z) = e^{i\alpha}/(1 - z\bar{z}_0)$, we see that

$$w'(z_0) = \frac{e^{i\alpha}}{1 - |z|^2},$$

i.e. $\arg w'(z_0) = \alpha$. This means that the angle of rotation of curves at point z_0 under the mapping (12.10) is α.

By Riemann's theorem, there is a unique mapping $w = w(z)$ that maps the unit circle $|z| < 1$ onto the unit circle $|w| < 1$ and satisfies the following conditions: $w(z_0) = 0$ and $\arg w'(z_0) = \alpha$. Consequently, every linear-fractional mapping of the unit circle $|z| < 1$ onto the unit circle $|w| < 1$ has the form (12.10).

Example 14. Find the function $w = f(z)$ that maps a unit circle conformally onto itself and such that

$$f\left(\frac{i-1}{2}\right) = 0, \quad \arg f'\left(\frac{i-1}{2}\right) = \frac{\pi}{2}.$$

Solution. The function

$$w = e^{i\alpha} \frac{z - z_0}{1 - z\bar{z}_0}$$

(with α arbitrary but real) maps the unit circle $|z| < 1$ onto the unit circle $|w| < 1$, so that point $z_0 = (i - 1)/2$ is mapped into the center $w = 0$. We have

$$w = f(z) = e^{i\alpha} \frac{z - \frac{i-1}{2}}{1 + z\frac{1+i}{2}}, \quad \text{or} \quad f(z) = e^{i\alpha} \frac{2z + 1 - i}{2 + z(1+i)}.$$

Since

$$f'(z) = e^{i\alpha} \frac{2}{[2 + (1+i)z]^2},$$

we have

$$f'(z_0) = f'\left(\frac{i-1}{2}\right) = 2e^{i\alpha}.$$

According to the condition $\arg f'\left(\frac{i-1}{2}\right) = \pi/2$ we have $\arg(2e^{i\alpha}) = \pi/2$, whence $\alpha = \pi/2$ and

$$f(z) = i\frac{2z+1-i}{2+(1+i)z}, \text{ or } f(z) = \frac{1-(2z+1)i}{2+z(1+i)}.$$

460. Find the linear-fractional function that maps the real axis onto the unit circle (circumference).

Find the images of the following domains for the given linear-fractional mappings:

461. The annulus $1 < |z| < 2$ for $w = (z-1)/(z+2)$.

462. The exterior of the unit circle, $|z| > 1$, for $w = (z+i)/(z-i)$.

463. The circle $|z| < 1$ for $w = (z-1)/(z+i)$.

464. Find the image of the interior of the circle $|z| < 1$ obtained by means of a linear-fractional function that maps the points $z_1 = 1$, $z_2 = i$, and $z_3 = \infty$ into the points $w_1 = 0$, $w_2 = \infty$, $w_3 = 1$.

465. Find the points that are symmetric to point $z = 1+i$ with respect to the following curves:

(a) $x = 0$; (b) $|z| = \sqrt{2}$; (c) $|z-1-i| = 2$.

466. Find the general form of the linear-fractional function that maps

(a) the upper half-plane onto the lower half-plane;

(b) the upper half-plane onto the right half-plane.

467. Find the mapping of the upper half-plane onto itself with

$$w(0) = 1, \quad w(1) = 2, \quad w(2) = \infty.$$

468. Find the mapping of the upper half-plane $\operatorname{Im} z > 0$ onto the unit circle $|w| < 1$ with the following properties:

(a) $w(i) = 0$, $\arg w'(i) = -\pi/2$;

(b) $w(2i) = 0$, $\arg w'(2i) = 0$.

469. Find the function $w = f(z)$ that maps the upper half-plane onto the unit circle $|w| < 1$ so that the points $z_1 = -1$, $z_2 = 0$, and $z_3 = 1$ are mapped into the points $w_1 = 1$, $w_2 = i$, and $w_3 = -1$ on the circumference.

470. Find the function $w = f(z)$ that maps the circle $|z| < 1$ onto the lower half-plane in such a way that the points 1, i, and $-i$ are mapped into the points 1, 0, and -1, respectively.

471. Find the linear-fractional function that maps the circle $|z| < 1$ onto the upper half-plane $\operatorname{Im} z > 0$ in such

a way that the points -1, 1, and i are mapped into the points ∞, 0, and 1, respectively.

472. Find the conformal mapping of the circle $|z| < 5$ onto the circle $|w| < 1$ such that the points -5, $4 + 3i$, and 5 are mapped into the points -1, i, and 1.

473. Find the function $w = f(z)$ that conformally maps the unit circle onto itself and such that

(a) $f(1/2) = 0$, $\arg f'(1/2) = \pi/2$;

(b) $f(0) = 0$, $\arg f'(0) = -\pi/2$.

474. Find the linear-fractional function that maps the circle $|z-2| < 3$ onto the circle $|w|<1$ in such a way that the points -1, 5, and $i\sqrt{5}$ are mapped into the points 1, i, and -1, respectively.

475. Onto what domain in the complex w plane does the function $w = i\frac{1-z}{1+z}$ map the upper semicircle $|z| < 1$, $\operatorname{Im} z > 0$?

476. Find the image of the domain D: $1 \leqslant |z| \leqslant 2$, $0 \leqslant \arg z \leqslant \pi/4$ obtained via the mapping $w = 1 + 1/z$.

Conformal mappings performed by the basic elementary functions

1. The *power function*

$$w = z^n, \tag{12.11}$$

with $n \geqslant 2$ a positive integer. The mapping performed by the power function is conformal in the entire complex plane except at point $z = 0$; while for $z \neq 0$ we have $w' = nz^{n-1}$, for $z = 0$ we have $w' = 0$. At $z = 0$ the conformality is violated, since under the mapping (12.11) all angles increase n-fold. The function (12.11) maps the angle $0 < \varphi < 2\pi/n$ in a one-to-one manner onto the entire complex plane with a cut along the real positive semiaxis, with the ray $\varphi = 0$ corresponding to the upper bank of the cut and the ray $\varphi = 2\pi/n$ to the lower bank. We obtain the same mapping for each angle into which the rays $\varphi = 2k\pi/n$ (k an integer) divide the complex z plane, and when $2(k-1)\pi/n < \varphi < 2k\pi/n$ ($k = 1, 2, \ldots, n$) is mapped onto the plane with the cut, the ray $\varphi = 2(k-1)\pi/n$ corresponds to the upper bank and the ray $\varphi = 2k\pi/n$ to the lower.

Example 15. Map the sector $0 < \arg z < \pi/4$ onto the unit circle $|w| < 1$ in such a way that point $z_1 = e^{i\pi/8}$ is mapped into the center $w_1 = 0$ and point $z_2 = 0$ into point $w_2 = 1$.

Solution. We use the function $t = z^4$ to map the sector $0 < \arg z < \pi/4$ (Fig. 19*a*) onto the upper half-plane

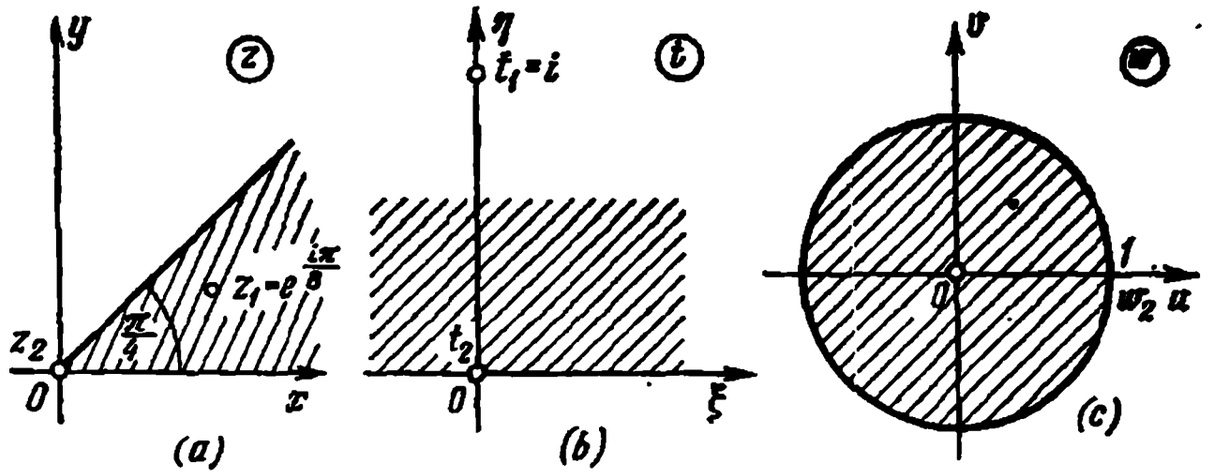


Figure 19

$\operatorname{Im} t > 0$ (Fig. 19*b*). The point $z_1 = e^{i\pi/8}$ is mapped into the point $t_1 = z_1^4 = i$, and $z_2 = 0$ is mapped into $t_2 = 0$. We then map the half-plane $\operatorname{Im} t > 0$ onto the circle $|w| < 1$ in such a way that point $t_1 = i$ is mapped into the center of the circle (Fig. 19*c*). Employing (12.9), we find that

$$w = e^{i\varphi} \frac{t - i}{t + i}.$$

The requirement that $t_2 = 0$ be mapped into $w_2 = 1$ yields $e^{i\varphi} = -1$. Substituting this result and $t = z^4$ into the above expression for w, we finally obtain

$$w = -\frac{z^4 - i}{z^4 + i}.$$

Example 16. Find the function that maps the upper semicircle $|z| < 1$, $\operatorname{Im} z > 0$ onto the upper half-plane $\operatorname{Im} w > 0$.

Solution. The given domain is a lune with its vertices at $z_1 = -1$ and $z_2 = 1$ (the angle at each vertex is $\alpha = \pi/2$) (Fig. 20*a*). The auxiliary function $t = (1 + z)/(1 - z)$ maps this lune conformally onto the first quadrant of the

complex t plane (Fig. 20*b*). Then either $w = t^2$, or $w = \left(\frac{1+z}{1-z}\right)^2$, yields the required map (Fig. 20*c*).

2. The function $w = \sqrt[n]{z}$. This is the inverse of the power function $z = w^n$, and for each $z = \rho e^{i\varphi}$ ($z \neq 0$ and $z \neq \infty$) the function has n different values given by the formula

$$w_k = \sqrt[n]{\rho}\left(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right),$$

$$k = 0, 1, \ldots, n-1.$$

Each of the functions w_k is a branch of the multiple-valued function $w = \sqrt[n]{z}$, and point $z = 0$ is an (algebraic)

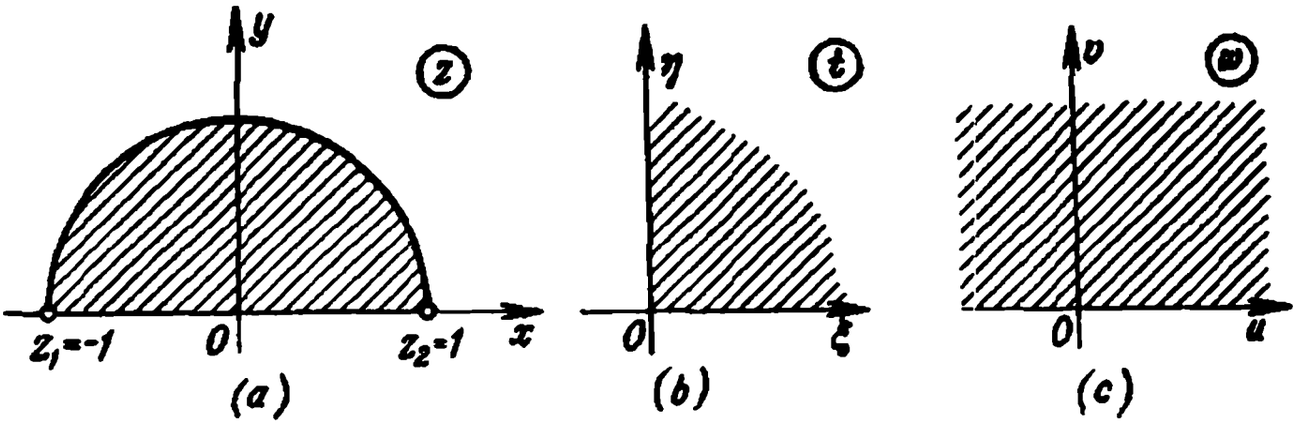


Figure 20

branch point of this function (the other branch point is $z = \infty$). In the extended complex z plane with any cut from $z = 0$ to $z = \infty$, for one, with a cut that is along the real positive semiaxis, we can separate n branches w_k of the function $w = \sqrt[n]{z}$. These branches univalently map the extended complex plane with a cut along the real positive semiaxis onto sectors

$$\frac{2(k-1)\pi}{n} < \arg w < \frac{2k\pi}{n}, \quad (k = 1, 2, \ldots, n).$$

Example 17. Map the upper half-plane $\operatorname{Im} z > 0$ with a cut from point $z_1 = 0$ to point $z_2 = ia$ $(a > 0)$ into the upper half-plane $\operatorname{Im} w > 0$ (in short, eliminate the cut!).

Solution. (1) Use the mapping $t = z^2$ to double the angles at the origin of coordinates. The segment z_1z_2 is then mapped into the segment t_1t_2 in the extended complex t plane, with $t_1 = 0$ and $t_2 = -a^2$, while the ray $\arg z = \pi$ (the real negative semiaxis) is mapped into the ray $\arg w = 2\pi$.

Thus, the initial domain has been mapped onto the extended complex t plane with a cut from point $t_2 = -a^2$ to the point $t = +\infty$.

(2) Use the linear function $\tau = t + a^2$ to shift the cut to the origin of coordinates. The extended complex τ plane proves to be cut from point $\tau = 0$ to point $\tau = +\infty$.

(3) Finally, use the function $w = \sqrt{\tau}$ (to be more precise, its branch that assumes positive values on the upper bank of the cut) to map the cut complex τ plane onto the half-plane $\operatorname{Im} w > 0$. Thus,

$$w = \sqrt{\tau} = \sqrt{t + a^2} = \sqrt{z^2 + a^2}.$$

Find the functions that map the following domains onto the upper half-plane:

477. The plane with a cut along the segment $[-1, 1]$.

478. The strip $0 < x < 1$ with a cut along the ray $x = 1/2$, $a \leqslant y < \infty$ $(a > 0)$.

479. The plane with cuts along the rays $y = 0$, $-\infty < x \leqslant a$ and $y = 0$, $b \leqslant x < +\infty$ $(a < b)$.

3. The *exponential function.* The mapping performed by the exponential function

$$w = e^z$$

is conformal in the entire complex plane, since $w' = e^z \neq 0$ at any finite point in the complex z plane.

If we divide the complex z plane into strips

$$2k\pi < y < 2(k+1)\pi, \quad (k = 0, \pm 1, \pm 2, \ldots),$$

each strip will be mapped by the function $w = e^z$ in a one-to-one manner onto the entire complex w plane with a cut along the real positive semiaxis. We assume that the upper bank of the cut corresponds to the lower edge $y = 2k\pi$ of a strip and the lower bank to the upper edge $y = 2(k+1)\pi$ of the same strip. In the process the points $z_0 = x_0 + iy_0$ and $z_k = x_0 + i(y_0 + 2k\pi)$ $(k = \pm 1, \pm 2, \ldots)$ are mapped into one point in the complex w plane. This means that the exponential function is an infinitely sheeted periodic function of complex variable z with the pure imaginary period $2\pi i$. The univalence domain of this function is any strip $y_0 \leqslant y < y_0 + 2\pi$, which is mapped in the entire complex w plane with a cut along the ray $\arg w_0 = y_0$ (Fig. 21). Note that the exponential function $w = e^z$ vanishes nowhere in the complex z plane.

Example 18. What is the result of mapping the half-strip

$$0 < \operatorname{Im} z < 2\pi, \quad \operatorname{Re} z < 0$$

via the function $w = e^z$?

Solution. We put $z = x + iy$ and $w = e^z = e^x e^{iy} = \rho e^{i\varphi}$. Then

$\rho = e^x$ and $\varphi = y$, where $-\infty < x < 0$, $0 < y < 2\pi$, so that $0 < \rho < 1$, $0 < \varphi < 2\pi$. Obviously, the points

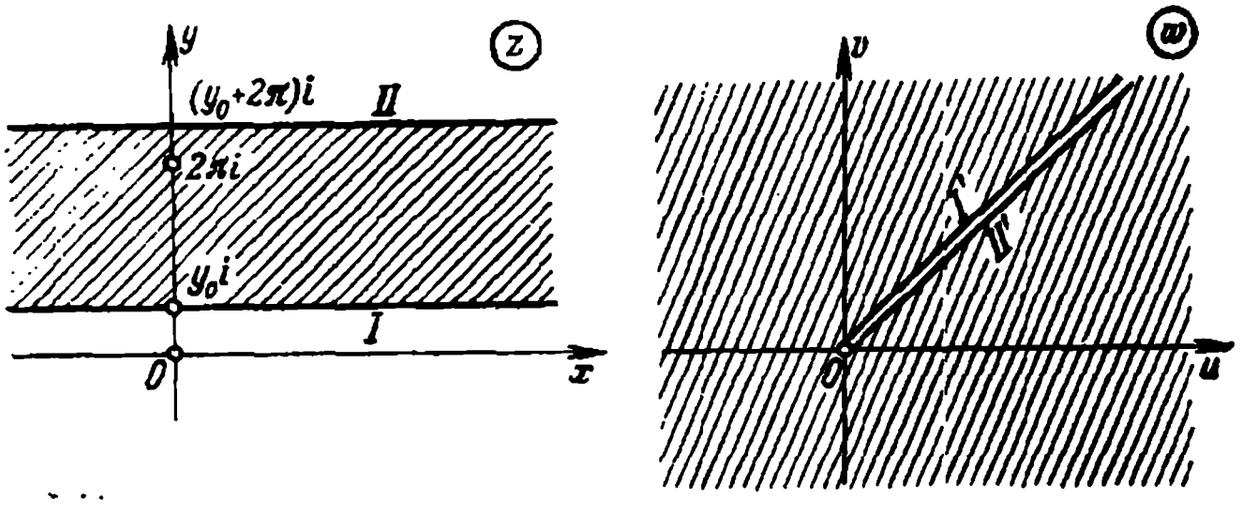


Figure 21

$w = \rho e^{i\varphi}$ satisfying such conditions occupy the circle $|w| < 1$ with a cut along the segment connecting points $w = 0$ and $w = 1$. Indeed, let us traverse the boundary γ of D in the positive sense starting from section I ($-\infty < x < 0$, $y = 0$) through section II ($x = 0$, $0 < y < 2\pi$) through section III ($y = 2\pi$ and x varies from 0 to $-\infty$). Obviously, these sections have corresponding sections in the complex w plane, I', II', and III', where I' coincides with the upper bank of the cut and III' with the lower (Fig. 22).

4. The *logarithmic function*

$$w = \operatorname{Ln} z$$

is defined as the inverse of the exponential function. For the sake of definiteness we take the principal value of the logarithm of z, i.e. that of its values which corresponds to the principal value of the argument:

$$\ln z = \ln |z| + i \arg z, \quad -\pi < \arg z \leqslant \pi.$$

This function is analytic at all finite points $z \neq 0$, and $w' = 1/z \neq 0$. This means that a mapping performed by the function $w = \ln z$ is conformal at all such points. Note that the points $z = 0$ and $z = \infty$ are the branch points of the function $w = \operatorname{Ln} z$, with $\operatorname{Ln} 0 = \infty$ and $\operatorname{Ln} \infty = \infty$.

No finite number of circuits (in the same direction) about point $z = 0$ will bring us to the initial branch of $\operatorname{Ln} z$ (chosen arbitrarily). Such branch points are called *logarithmic*.

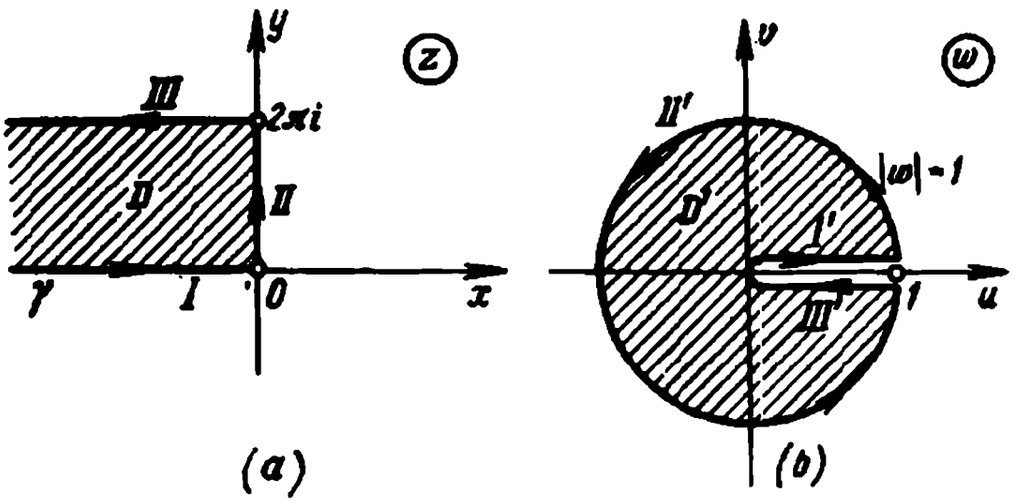


Figure 22

Example 19. Find the function that maps the complex z plane cut along the real axis from point $z = 0$ to point $z = -\infty$ onto the strip $-\pi < v < \pi$ in the complex w plane.

Solution. When we studied the exponential function $w = e^z$, we noted (see p. 154) that any strip $y_0 \leqslant y < y_0 + 2\pi$ is mapped by this function into the entire complex w plane with a cut along the ray $\arg w_0 = y_0$. Let us consider the inverse mapping, i.e. of a strip $v_0 \leqslant v < v_0 + 2\pi$, where $v_0 = -\pi$, in the complex w plane onto the entire complex z plane with a cut along the ray $\arg z_0 = v_0 = -\pi$ (Fig. 23). Such a mapping, obviously, is performed by the function $z = e^w$; consequently, the sought mapping is $w = \ln z = \ln |z| + i \arg z$. When point z runs along the lower bank of the cut, I, from $x = -\infty$ to $x = 0$, the corresponding point in the complex w plane runs along the line I' from $u = +\infty$ to $u = -\infty$ ($v = -\pi$). Next, when point z runs along the upper bank of the cut, II, from $x = 0$ to $x = -\infty$, the corresponding point in the complex w

plane runs along the line II' from $u = -\infty$ to $u = +\infty$ ($v = \pi$), so that the domain D and the corresponding domain G remain to the right.

5. The *trigonometric functions*

$$w = \sin z, \quad w = \cos z.$$

For every complex-valued z we have

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Example 20. Into what does the function $\cos z$ map the half-strip $0 < x < \pi$, $y > 0$ (Fig. 24)?

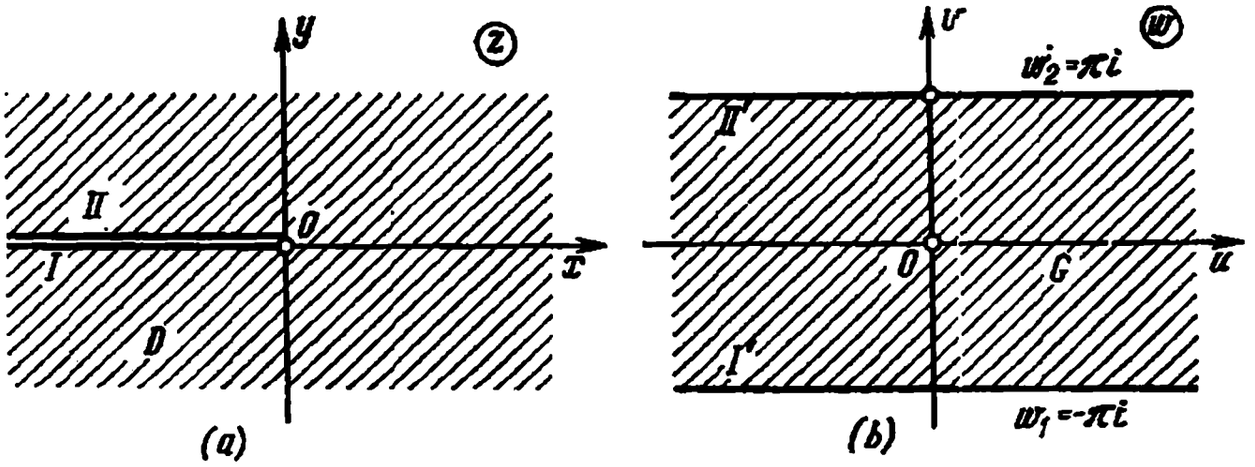


Figure 23

Solution. We have $\cos z = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$. When point z runs along part I of the boundary from $y = +\infty$ to $y = 0$ (with $x = 0$), the corresponding point in the complex w plane runs along section I' from $u = +\infty$ to $u = 1$ (with $v = 0$). When point z runs along II from $x = 0$ to $x = \pi$ (with $y = 0$), $w = \cos x$ runs along II' from $u = 1$ to $u = -1$ (with $v = 0$). Finally, when point z runs along III from $y = 0$ to $y = +\infty$ (with $x = \pi$), $w = \cosh y$ runs along III' from $u = -1$ to $u = -\infty$ (with $v = 0$). Thus, when point z traverses the boundary of the half-strip $0 < x < \pi$, $y > 0$ in such a way that the half-strip remains to the left, point w runs along the entire real axis and therefore, by the principle of one-to-one correspondence between boundaries, the function $w = \cos z$ maps the half-strip we are considering here onto the lower half-plane $\operatorname{Im} w > 0$. We can similarly show that

the same function maps the half-strip $0 < x < \pi$, $y < 0$ onto the upper half-plane $\operatorname{Im} w > 0$. The edge $x = \pi$, $y < 0$ of this half-strip corresponds to the segment $-\infty < u < -1$ of the real axis in the complex w plane, the edge $0 < x < \pi$, $y = 0$ (traversed from π to 0) corresponds to the segment $-1 < u < 1$, and the edge $x = 0$, $y < 0$ to the segment $1 < u < +\infty$. The segment $-\infty < u < -1$ of the real axis in the complex w plane is traversed twice,

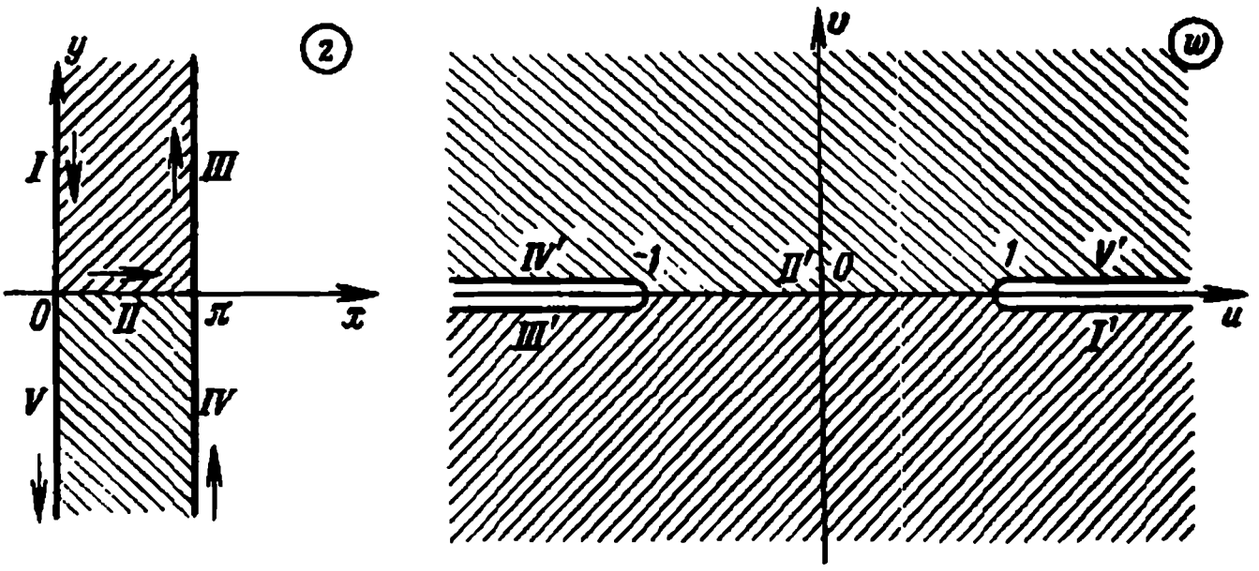


Figure 24

namely, both the edge $x = \pi$, $y > 0$ of the first half-strip and the edge $x = \pi$, $y < 0$ of the second are mapped onto it. For the mapping performed by $w = \cos z$ to be one-to-one we must cut the complex w plane along the real axis from $-\infty$ to -1 (and from 1 to $+\infty$). Therefore, the function $w = \cos z$ maps the strip $0 < x < \pi$ onto the entire complex w plane with cuts along the real axis from $-\infty$ to -1 and from 1 to $+\infty$.

Example 21. Map a unit circle (disk) with a cut along the real axis starting at the center onto the upper half-plane.

Solution. (1) We use the function $w_1 = \sqrt{z}$ to map the unit circle (with the cut) onto the upper semicircle. In the process the upper bank OA of the cut, i.e. the segment $[-1, 0]$, remains in place, while the lower bank OA' is mapped into the segment $[-1, 0]$ in the complex W_1 plane.

(2) Next we use the function

$$w_2 = \frac{w_1 + 1}{w_1 - 1}$$

to map the above semicircle onto the first quadrant of the complex W_2 plane. The point $w_1 = 1$ will be mapped into point $w_2 = \infty$ and the point $w_1 = -1$ into point $w_2 = 0$.

(3) Finally, we use the function $w = w_2^2$ to map the first quadrant onto the upper half-plane. The final result

$$w = w_2^2 = \left(\frac{w_1+1}{w_1-1}\right)^2 = \left(\frac{\sqrt{z}+1}{\sqrt{z}-1}\right)^2.$$

Example 22. Find the function that maps the domain between two circles that intersect at real points $z = a$ and $z = b$ $(a < b)$ at an angle $\pi\lambda$, $0 < \lambda < 1$ (Fig. 25),

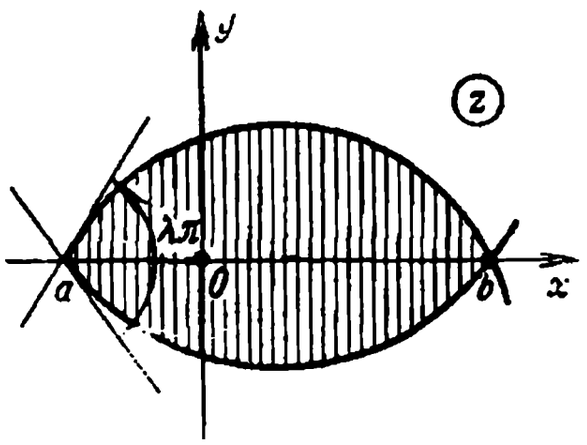


Figure 25

onto the upper half-plane $\operatorname{Im} w > 0$ in such a way that point $z = a$ is mapped into point $w = 0$ and point $z = b$ into point $w = \infty$.

Solution. (1) We map the lune ab into an angle $\lambda\pi$, $0 < \lambda < 1$, in such a way that point $z = a$ is mapped into point $w = 0$ and point $z = b$ into point $w = \infty$. This is performed by the function $w_1 = (z - a)/(b - z)$ (Fig. 26).

(2) The mapping $w_2 = w_1 e^{i\alpha}$ rotates the angle V_1 of magnitude $\lambda\pi$ counterclockwise through the angle α. Thus, in the complex W_2 plane we have an angle V_2 of the same magnitude as angle V_1 (Fig. 27).

(3) The function $w = w_2^{1/\lambda}$ maps the angle V_2 into a half-plane. Thus, the sought mapping is

$$w = c\left(\frac{z-a}{b-z}\right)^{1/\lambda},$$

where c is a complex-valued constant chosen in such a way that the mapping is onto the upper half-plane.

Example 23. Find the function that conformally maps the domain D: $|z| < 2$, $|z-1| > 1$ (Fig. 28) onto the upper half-plane.

Solution. (1) The function

$$w_1 = \frac{z}{z-2}$$

maps the given domain into the strip Π_1: $\{0 < u < 1/2,$

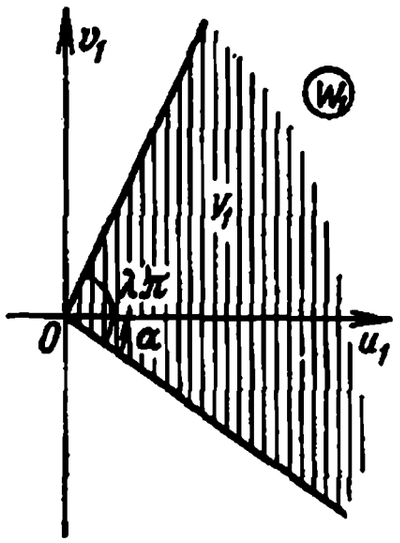


Figure 26

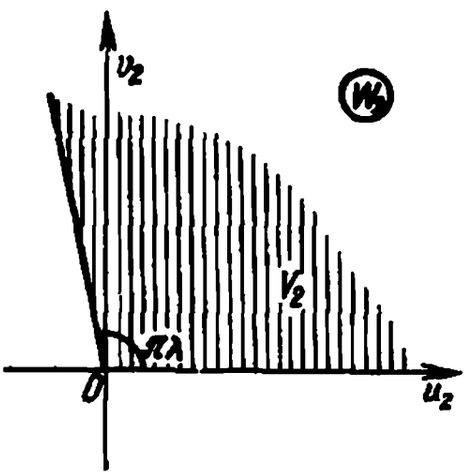


Figure 27

$-\infty < v < +\infty\}$. Indeed, if we put $z = 2e^{i\varphi}$, we obtain

$$w_1 = u + iv = \frac{2e^{i\varphi}}{2e^{i\varphi}-2} = \frac{e^{i\varphi}}{e^{i\varphi}-1} \quad (0 \leqslant \varphi < 2\pi).$$

Separating the real and imaginary parts, we obtain

$$u = \frac{1}{2}, \quad v = -\frac{\cos\frac{\varphi}{2}}{2\sin\frac{\varphi}{2}}.$$

Thus, the function $w_1 = z/(z-2)$ maps the circle $|z| = 2$ into the straight line

$$u = \frac{1}{2}, \quad v = -\frac{\cos\frac{\varphi}{2}}{2\sin\frac{\varphi}{2}},$$

traversed upward (Fig. 29). For one, the points $z_1 = 2i$, $z_2 = -2$, and $z_3 = -2i$ are mapped into the points $w = (1-i)/2$, $w = 1/2$, and $w = (1+i)/2$. The same function maps the circle $|z-1| = 1$ into the straight line

$u = 0$, $-\infty < v < +\infty$. Indeed, substituting $1 + e^{i\varphi}$ for z in w_1, we find that

$$w_1 = u_1 + iv_1 = \frac{e^{i\varphi}+1}{e^{i\varphi}-1} = -i\,\frac{\cos\frac{\varphi}{2}}{\sin\frac{\varphi}{2}},$$

whence

$$u_1 = 0, \quad v_1 = -\frac{\cos\frac{\varphi}{2}}{\sin\frac{\varphi}{2}},$$

where φ varies between 2π and 0. This means that when point z runs along the circle γ_2, the corresponding point w_1

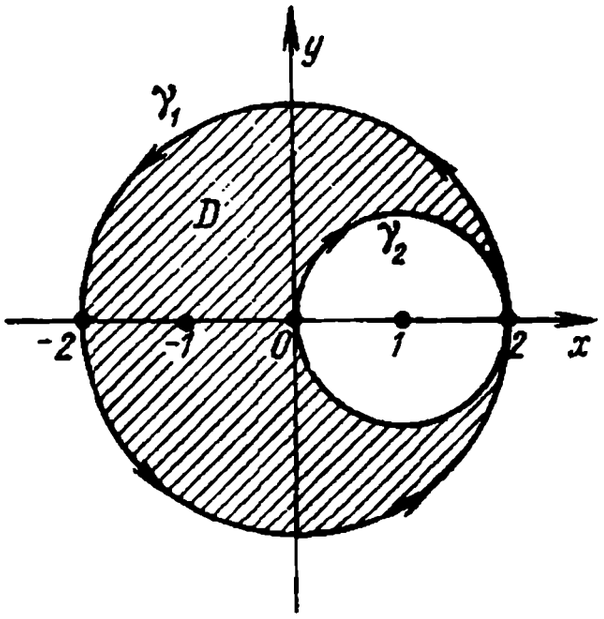


Figure 28

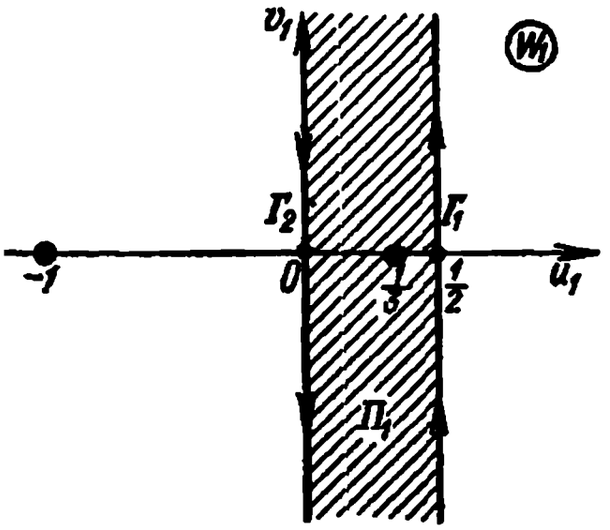


Figure 29

runs along the straight line Γ_2 (the v_1 axis in the complex W_1 plane) downward. Under the mapping $w_1 = z/(z-2)$ the interior point $z = -1$ of D is mapped into the interior point $w_1 = 1/3$ of Π_1 (Fig. 29). Thus, the function $w_1 = z/(z-2)$ maps D onto Π_1.

(2) The function $w_2 = 2\pi i w_1$ maps the vertical strip

$$\Pi_1 : \left\{0 < u_1 < \frac{1}{2},\ -\infty < v_1 < +\infty\right\}$$

into the horizontal strip

$$\Pi_2 : \{-\infty < u_2 < +\infty,\ 0 < v_2 < \pi\} \text{ (Fig. 30).}$$

(3) The function $w = e^{w_2}$ maps Π_2 into the upper half

of the complex W plane. The final result is

$$w = e^{w_2} = e^{2\pi i w_1} = e^{2\pi i \frac{z}{z-2}}.$$

6. The *Zhukovskii (Joukowski) function*

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right), \tag{12.12}$$

named after N. E. Zhukovskii, a Russian applied mathematician and aerodynamicist who was the first to use it widely in aerodynamics, is analytic in the entire complex z

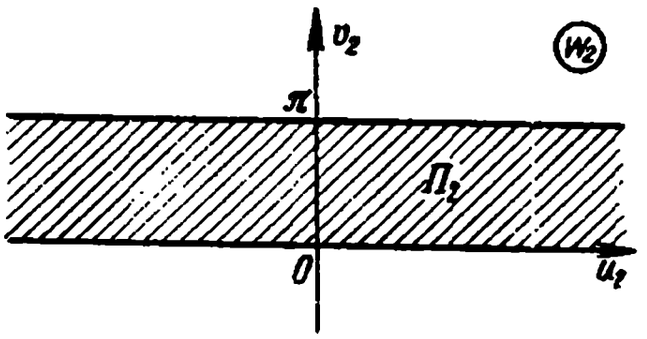


Figure 30

plane except at point $z = 0$, where it has a pole. Its derivative $w' = (1/2)(1 - 1/z^2)$ is nonzero everywhere except at the points $z = \pm 1$. Therefore, the mapping performed by the Zhukovskii function is comformal at all finite points except $z = -1$, $z = 0$, and $z = 1$. It is univalent in a domain that does not contain points z_1 and z_2 coupled by the condition $z_1 \times z_2 = 1$. The unit circle $|z| < 1$ and its exterior are examples of such domains.

Example 24. Show that the function $w = (1/2)(z + 1/z)$ conformally maps the unit circle $|z| < 1$ onto the entire complex w plane with a cut along the segment $[-1, 1]$ of the real axis, with the upper semicircle ($|z| = 1$, $\operatorname{Im} z > 0$) mapped onto the lower bank of the cut and the lower semicircle ($|z| \neq 1$, $\operatorname{Im} z < 0$) onto the upper bank.

Solution. We put $w = u + iv$ and $z = \rho e^{i\varphi}$, where $\rho < 1$. Then (12.12) takes the form

$$u + iv = \frac{1}{2}\left(\rho e^{i\varphi} + \frac{1}{\rho e^{i\varphi}}\right)$$

$$= \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)\cos\varphi + i\frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\sin\varphi,$$

and, consequently,

$$u = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)\cos\varphi, \quad v = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\sin\varphi. \qquad (12.13)$$

If we now exclude φ, we arrive at the equation of an ellipse

$$\frac{u^2}{\frac{1}{4}\left(\rho+\frac{1}{\rho}\right)^2} + \frac{v^2}{\frac{1}{4}\left(\rho-\frac{1}{\rho}\right)^2} = 1,$$

with semiaxes

$$a = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right), \quad b = \frac{1}{2}\left|\rho - \frac{1}{\rho}\right|$$

and foci at points (—1, 0) and (1, 0) (for all values of $\rho < 1$). The two formulas in (12.13) show that as point z moves along the circle $|z| = \rho$ in the positive sense, point w moves along an ellipse in the negative sense, since $\rho - 1/\rho < 0$ for $\rho < 1$. Obviously, different circles correspond to different confocal ellipses. As $\rho \to 1 - 0$,

$$u \to \cos\varphi, \quad v \to 0, \quad a \to 1, \quad b \to 0,$$

i.e. the ellipses flatten out without limit and their limiting position (corresponding to the circle $\rho = 1$) is the double segment [—1, 1]. As shown above, the upper semicircle of $|z| = 1$ is mapped onto the lower bank of the segment [—1, 1] traversed from 1 to —1, while the lower semicircle is mapped onto the upper bank traversed from —1 to 1 (Fig. 31).

Similarly, the Zhukovskii function maps the domain $|z| > 1$ into the second copy of the complex w plane with a cut along the segment [—1, 1] of the real axis, where the upper semicircle $|z| = 1$, $\operatorname{Im} z > 0$ is mapped onto the upper bank of the cut and the lower semicircle $|z| = 1$, $\operatorname{Im} z < 0$ onto the lower bank (Fig. 31).

Every circle (circumference) of radius $R \neq 1$ is mapped via (12.12) onto an ellipse with semiaxes

$$a = \frac{1}{2}\left(R + \frac{1}{R}\right), \quad b = \frac{1}{2}\left|R - \frac{1}{R}\right|$$

and foci at points (—1, 0) and (1, 0). The rays $\arg z = \varphi$ (except $\varphi = 0, \pm\pi/2$, and π) are mapped onto corresponding branches of the hyperbola

$$\frac{u^2}{\cos^2\varphi} - \frac{v^2}{\sin^2\varphi} = 1;$$

the rays $\arg z = 0$, $\arg z = \pm\pi/2$, and $\arg z = \pi$ are mapped onto infinite segments of the real or imaginary axis traversed twice.

Example 25. Using the Zhukovskii function, find the image of the domain

$$0 < |z| < 1, \quad 0 < \arg z < \frac{\pi}{4}.$$

Solution. Substituting $z = re^{i\varphi}$ into the Zhukovskii

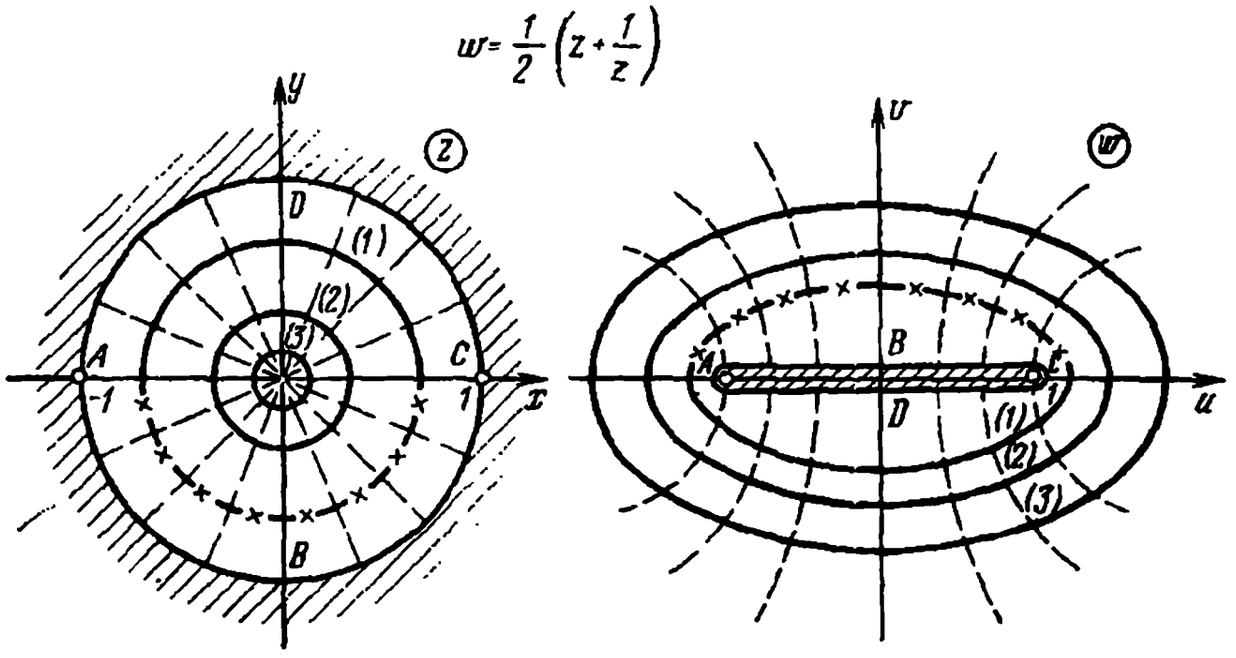


Figure 31

function

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

and separating the real and imaginary parts, we obtain

$$\begin{cases} u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\varphi, \\ v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\varphi. \end{cases}$$

Traversing the contour of the sector in the positive sense $OABO$ (Fig. 32), we find that the segment OA is mapped into an infinite segment of the real axis traversed from $u = +\infty$ to $u = 1$ (Fig. 33), the arc AB of the circle $|z| = 1$ is mapped into the segment $A'B'$ of the real axis, and

the segment BO is mapped into the curve

$$u=\frac{\sqrt{2}}{4}\left(r+\frac{1}{r}\right),\quad v=\frac{\sqrt{2}}{4}\left(r-\frac{1}{r}\right),$$

or $u^2 - v^2 = 1/2$ (a hyperbola). According to the principle of one-to-one correspondence between boundaries, the Zhu-

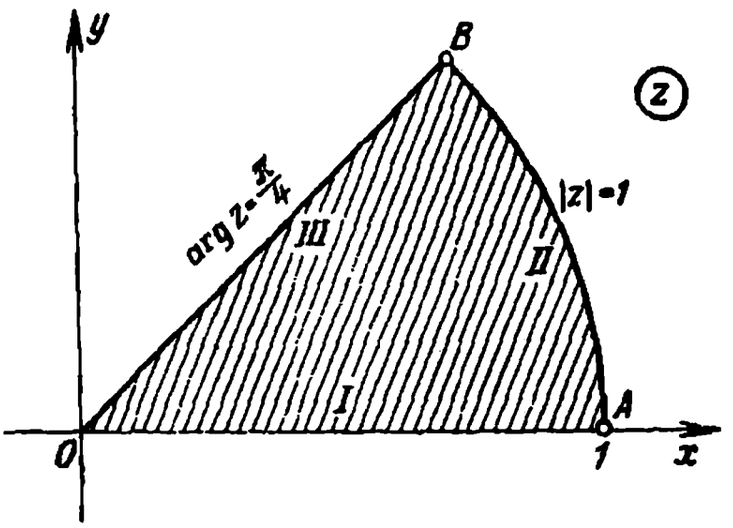


Figure 32

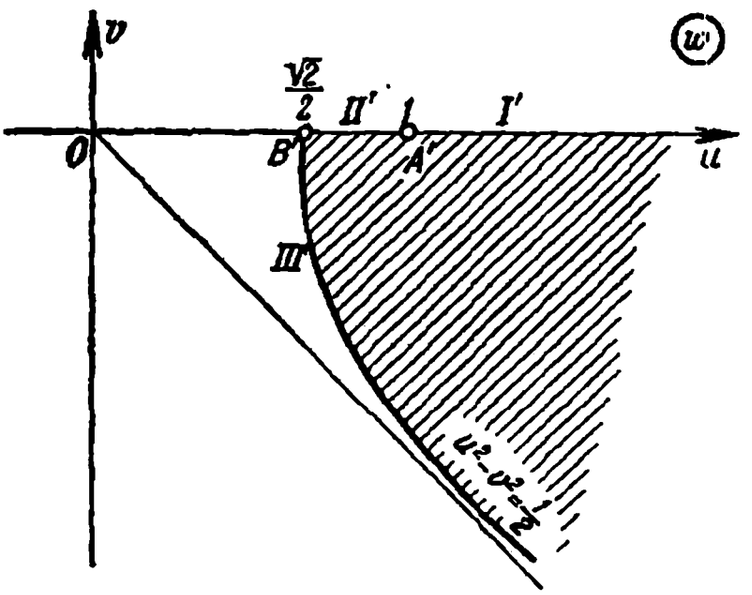


Figure 33

kovskii function maps the given sector into the domain

$$u^2-v^2>\frac{1}{2},\quad u>\frac{\sqrt{2}}{2},\quad v<0.$$

Example 26. Find the result of mapping the domain $\operatorname{Im} z > 0$ via the Zhukovskii function.

Solution. In the upper half-plane $\operatorname{Im} z > 0$ the mapping performed by (12.12) is conformal, since in this domain

there are no points z_1 and z_2 for which $z_1 \times z_2 = 1$. The function (12.12) maps the semicircles $|z| = \rho$, $\rho > 1$, $\operatorname{Im} z > 0$ into semiellipses that lie in the half-plane $\operatorname{Im} w > 0$ and cover it completely, while the semicircles $|z| = \rho$, $\rho < 1$, $\operatorname{Im} z > 0$ are mapped into semiellipses that lie

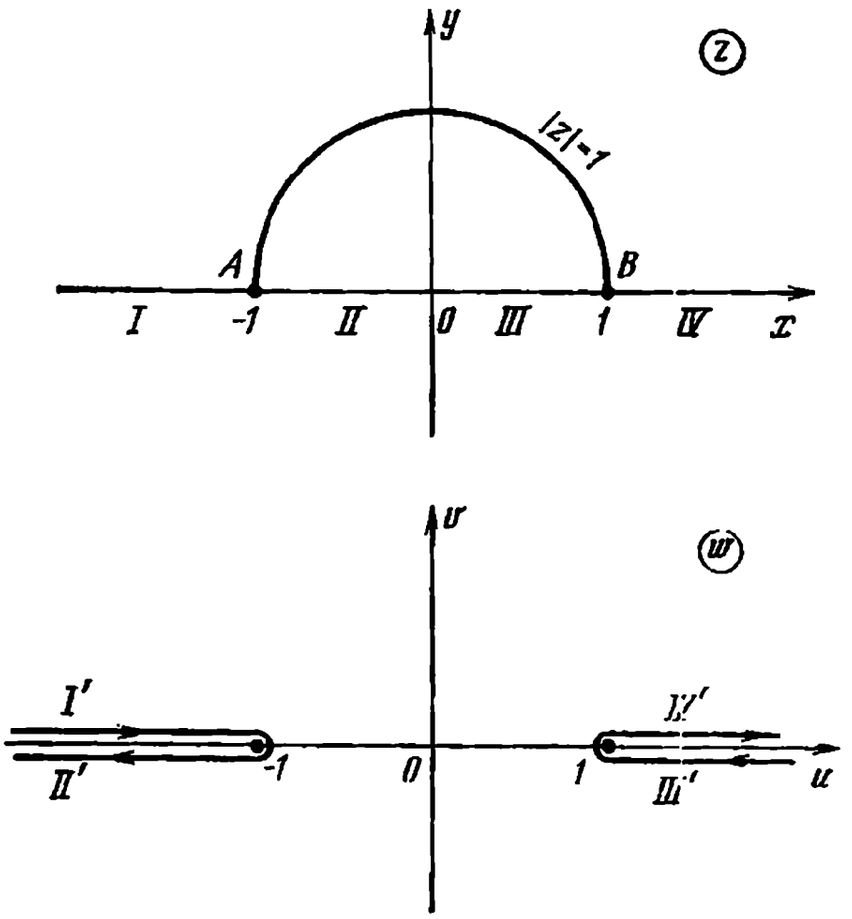


Figure 34

in the half-plane $\operatorname{Im} w < 0$ and cover it completely. The semicircle $|z| = 1$, $\operatorname{Im} z > 0$ is mapped into the segment $[-1, 1]$ of the real axis of the complex w plane. The boundary of $\operatorname{Im} z > 0$ is the real axis $(-\infty < x + \infty)$ and is mapped in the following manner (Fig. 34):

the infinite segments I $(-\infty, -1)$ and IV $(1, +\infty)$ are mapped into the segments I' and IV' in the complex w plane;

the segment II $(-1, 0)$ is mapped into II' $(-\infty, -1)$, while the segment III $(0, 1)$ is mapped into III' $(1, +\infty)$, with II' and III' traversed from right to left.

Thus, the domain $\operatorname{Im} z > 0$ is mapped onto the entire complex w plane with two infinite cuts, $(-\infty, -1)$ and $(1, \infty)$.

Example 27. Onto what domain in the complex w plane does the Zhukovskii function (12.12) map the domain D: $\operatorname{Im} z > 0$, $|z| > 1$?

Solution. We cover D with semicircles $|z| = \rho$, $\rho > 1$, $\operatorname{Im} z > 0$. The Zhukovskii function maps them onto ellipses that cover the upper half plane $\operatorname{Im} w > 0$. The semicircle $|z| = 1$ is mapped into the segment $[-1, 1]$, the ray

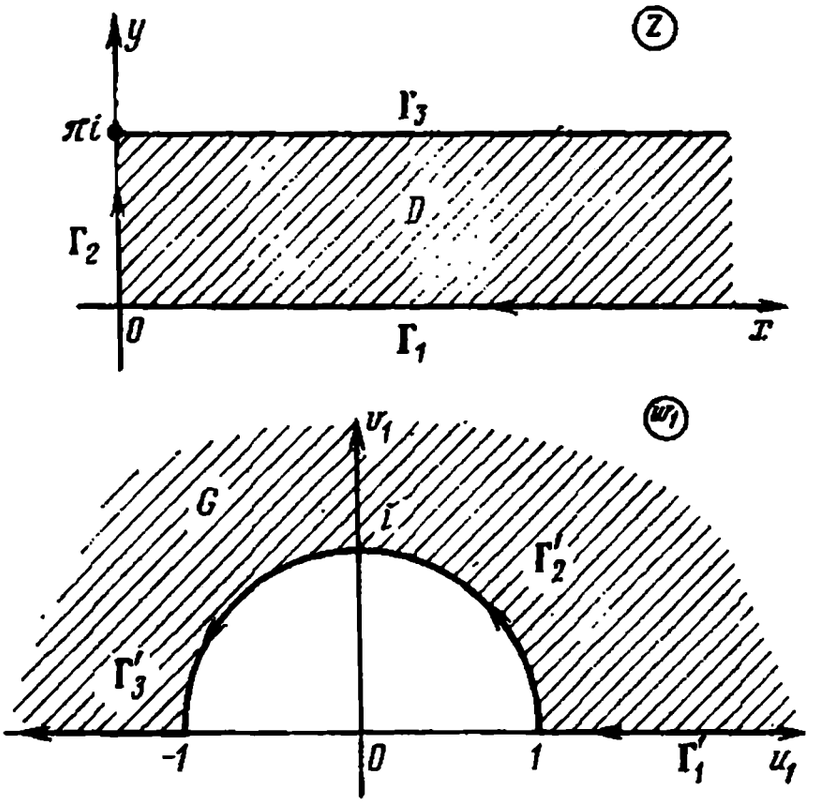


Figure 35

$(-\infty, -1]$ into the ray $(-\infty, -1]$, and the ray $[1, +\infty)$ into the ray $[1, +\infty)$. Thus, D is mapped into the upper half-plane $\operatorname{Im} w > 0$.

Example 28. Onto what domain in the complex w plane does the function $w = \cosh z$ map the domain D: $0 < \operatorname{Im} z < \pi$, $\operatorname{Re} z > 0$?

Solution. Since

$$w = \cosh z = \frac{e^z + e^{-z}}{2} = \frac{1}{2}\left(e^z + \frac{1}{e^z}\right),$$

by introducing the variable $w_1 = e^z$ we obtain

$$w = \frac{1}{2}\left(w_1 + \frac{1}{w_1}\right).$$

(a) The function $w_1 = e^z$ maps the half-strip D into the domain G, the exterior of a semicircle, $|w_1| > 1$, $\operatorname{Re} w_1 >$

> 0, and it is easy to see that the boundary $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ of D is mapped into the boundary $\Gamma' = \Gamma_1' + \Gamma_2' + \Gamma_3'$ of G (Fig. 35).

(b) Since

$$w = \frac{1}{2}\left(w_1 + \frac{1}{w_1}\right)$$

is the Zhukovskii function, it maps G into the upper half-plane $\operatorname{Im} w > 0$.

Example 29. Show that the function $w = \cos z$ conformally maps the half-strip $-\pi < \operatorname{Re} z < 0$, $\operatorname{Im} z > 0$ into the upper half-plane $\operatorname{Im} w > 0$.

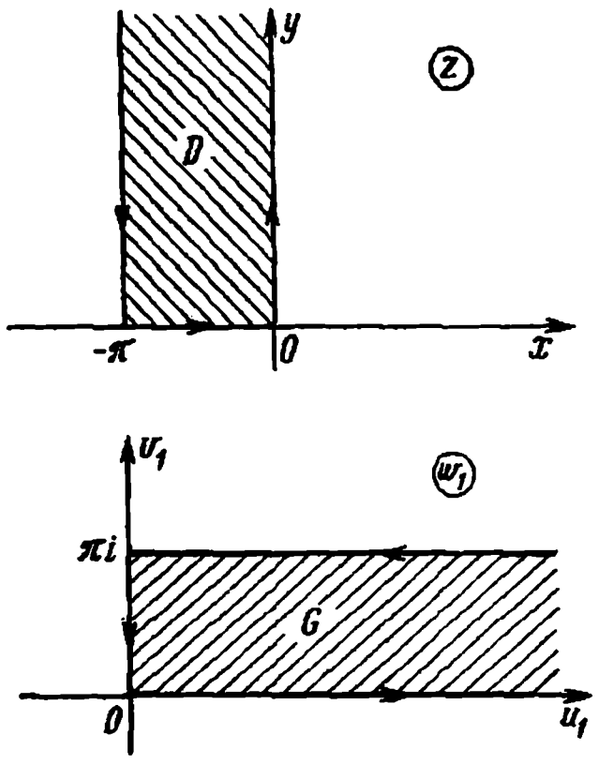


Figure 36

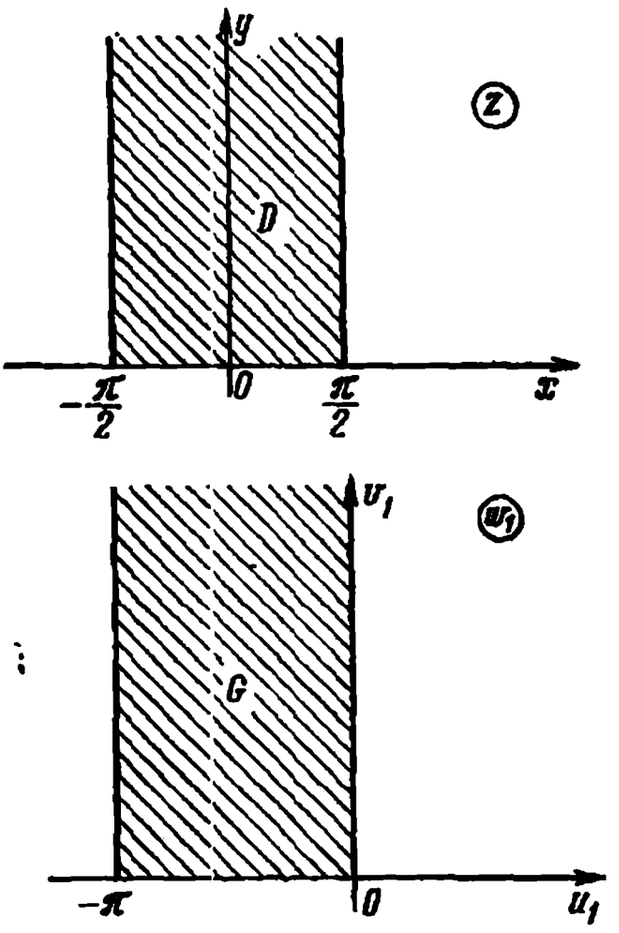


Figure 37

Solution. Since $\cos z = \cosh(-iz)$, introducing the variable $w_1 = -iz$ we obtain $w = \cos z = \cosh w_1$.

(a) The function $w_1 = -iz$ maps D: $-\pi < \operatorname{Re} z < 0$ into the half-strip G: $0 < \operatorname{Im} w_1 < \pi$, $\operatorname{Re} w_1 > 0$ (Fig. 36).

(b) In Example 28 we showed that the function $w = \cosh w_1$ maps the half-strip G into the upper half-plane $\operatorname{Im} w > 0$.

Example 30. Show that the function $w = \sin z$ maps the half-strip D: $-\pi/2 < \operatorname{Re} z < \pi/2$, $\operatorname{Im} z > 0$ onto the upper half-plane $\operatorname{Im} w > 0$.

Solution. Since $\sin z = \cos(z - \pi/2)$, introducing the notation $w_1 = z - \pi/2$ yields

$$w = \sin z = \cos w_1.$$

But the function $w_1 = z - \pi/2$ maps D into the half-strip G: $-\pi < \operatorname{Re} w_1 < 0$, $\operatorname{Im} w_1 > 0$ (Fig. 37), and the function $w = \cos w_1$ maps G into the upper half-plane $\operatorname{Im} w > 0$.

7. The function

$$z = w \pm \sqrt{w^2 - 1}, \tag{12.14}$$

which is the inverse of the Zhukovskii function. It is two-valued, and each of its branches maps the complex w plane with a cut from -1 to 1 along the real axis onto the interior or exterior of the unit circle in the complex z plane.

Example 31. Find the function that maps domain D, the half-plane $\operatorname{Re} z > 0$ with a cut from 0 to 1 along the real axis (Fig. 38.1), onto the half-plane $\operatorname{Re} w > 0$.

Solution. Inverting the mapping

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

and changing variables, we arrive at the function $w_1 = z + \sqrt{z^2 - 1}$, which maps the given domain D into the domain G shown in Fig. 38.2. (Here we take a specific branch of the root at $z > 1$.) Next, the linear-fractional function

$$w_2 = \frac{w_1 - i}{w_1 + i}$$

maps G into the fourth quadrant $\operatorname{Re} w_2 > 0$, $\operatorname{Im} w_2 < 0$ (Fig. 38.3). Finally the function $w = iw_2^2$ maps this quadrant into the right half-plane $\operatorname{Re} w > 0$. Therefore, the solution of the problem is given by the formula

$$w = i\left(\frac{z + \sqrt{z^2 - 1} - i}{z + \sqrt{z^2 - 1} + i}\right)^2.$$

480. Find the function that maps the unit circle onto the plane with a cut along the real positive semiaxis.

481. Find the function that maps the angle between the rays $z = z_0 + e^{i\varphi_1 t}$ and $z = z_0 + e^{i\varphi_2 t}$, with $t \geqslant 0$ and $0 < \varphi_1 < \varphi_2$, onto the upper half-plane.

482. Find the function that maps the horizontal strip $\{0 \leqslant x < +\infty,\ 0 \leqslant y \leqslant \pi\}$ onto the upper semicircle of $|z| < 1$.

483. Map the half-strip $\{0 \leqslant x < +\infty,\ 0 \leqslant y \leqslant \pi\}$ onto the upper half-plane.

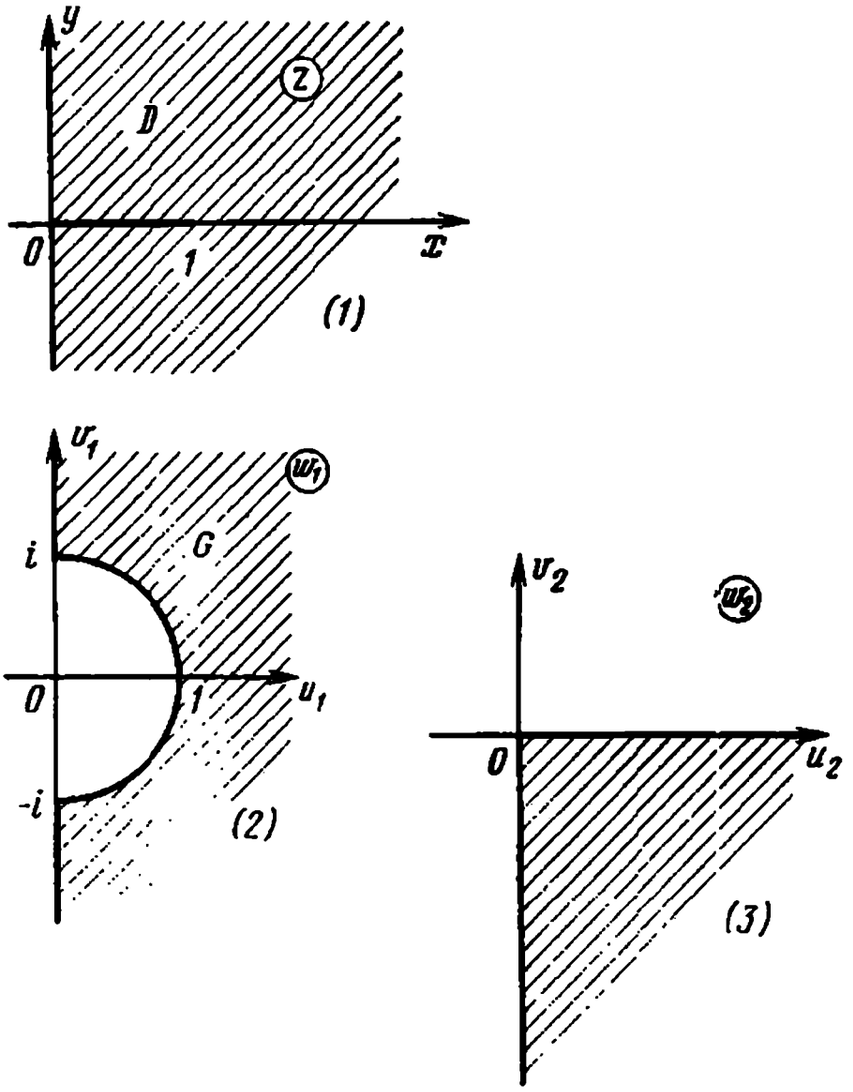


Figure 38

484. Onto what domain does the function $w = \ln z$ map the semiannulus $\{r \leqslant \rho \leqslant R,\ 0 \leqslant \varphi \leqslant \pi\}$? (The semiannulus is in the complex z plane.)

485. Find the function that maps the domain $\pi/2 < \arg z < \pi$ onto the domain $0 < \arg w < \pi/4$.

486. Map the complex z plane with a cut along the real axis, $0 < a < x < b$, onto the upper half of the complex w plane.

487. Map the complex z plane with straight cuts $(-\infty, b)$ and $(a, +\infty)$, with $b < a$ and real, onto the half-plane $\operatorname{Im} w > 0$.

488. Find the function that maps the first quadrant $0 \leqslant \arg z \leqslant \pi/2$ onto the circle $|w| < 1$ in such a way that the points $w = 0$ and $w = 1$ correspond to the points $z = 1 + i$ and $z = 0$, respectively.

In the problems below find the domains in the complex w plane onto which the functions $w = f(z)$ map the given domains D in the complex z plane:

489. $w = z^2 + 1$, D: a quarter of the unit circle, $|z| < 1$, $0 < \arg z < \pi/2$.

490. $w = e^{2z}$, D: the half-strip $0 < \operatorname{Im} z < \pi/4$, $\operatorname{Re} z < 0$.

491. $w = \ln z + 1$, D: the part of the annulus $1 < |z| < e$ that lies within the angle $0 < \arg z < e$.

Find the function that maps each of the domains given below onto the upper half-plane:

492. The sector $|z| < 2$, $0 < \arg z < \pi/4$.

493. The strip $a < \operatorname{Re} z < b$, $a > 0$.

Find the function that performs the following mapping:

494. The circle $|z| < 1$ with a cut along the radius from point $z = 0$ to point $z = -1$ onto the half-strip $-\pi < v < \pi$, $u < 0$.

495. The half-strip $0 < \operatorname{Im} z < \pi$, $\operatorname{Re} z > 0$ onto the half-plane $\operatorname{Im} w > 0$.

496. The strip $-\infty < \operatorname{Re} z < +\infty$, $0 < \operatorname{Im} z < \pi/2$ onto the complex w plane with cuts $-\infty < u \leqslant -1$, $v = 0$ and $1 \leqslant u < +\infty$, $v = 0$.

497. The lune within the circles $|z - 1| = 1$ and $|z - 2| = 2$ onto the strip $0 < \operatorname{Re} w < 1$.

498. Find the linear-fractional function that maps the domain D: $\{|z + 1| > 1;\ |z + 2| < 2\}$ onto the strip P: $\{0 < \operatorname{Im} w < 1\}$.

499. Find the function that maps the lune within the circles $|z - 1| = 1$ and $|z + i| = 1$ onto the half-plane $\operatorname{Re} w > 0$.

500. Use the Zhukovskii function to find the function that maps the annulus $1 < |z| \leqslant 2$ onto the domain $u^2/25 + v^2/9 \leqslant 1$ with the cut $-4 \leqslant u \leqslant 4$, $v = 0$.

501. Map the upper half-plane $\operatorname{Im} z > 0$ without the upper half of the circle $|z| < 1/2$ onto the circle $|w| < 1$.

502. Find the conformal mapping of the sector $0 < \arg z < \pi/8$ onto the unit circle $|w| < 1$ in such a way that point $z_1 = e^{i\pi/16}$ is mapped into the center $w_1 = 0$ and point $z = 0$ into point $w = 1$.

503. Map the strip $0 < x < \pi/4$ onto the first quadrant $0 < \arg w < \pi/2$.

504. Find the domains into which the function $w = \tan z$ maps

(a) the strip $-\pi/4 < \operatorname{Re} z < \pi/4$; (b) the strip $0 < \operatorname{Re} z < \pi$.

§ 13 The Complex-Valued Potential and Its Hydrodynamic Interpretation

Suppose we have a steady-state plane-parallel vector field

$$\mathbf{a} = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}, \qquad (13.1)$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the x and y (coordinate) axes. Since point (x, y) in the xOy plane and its radius vector $\mathbf{r} = x\mathbf{i} + y\mathbf{i}$ represent the same complex number $z = x + iy$, with i the unit imaginary number, the vector $\mathbf{a} = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ represents the complex number $P(x, y) + iQ(x, y)$. For this reason we can write (13.1) in another form

$$\mathbf{a} = P(x, y) + iQ(x, y).$$

This means that a vector field of the type (13.1) can be defined by fixing two real functions $P(x, y)$ and $Q(x, y)$ of two real variables x and y or one function of one complex variable z,

$$f(z) = P(x, y) + iQ(x, y).$$

Functions of one complex variable in hydrodynamics

Steady-state plane-parallel flow of a homogeneous incompressible fluid is completely defined if we know the analytic function $f(z) = u(x, y) + iv(x, y)$ called the *complex potential* or the *characteristic function* of the flow. The real part $u(x, y)$ and the imaginary part $v(x, y)$ of $f(z)$ are called, respectively, the *velocity potential* and the *stream function*. The curves $u(x, y) = \text{const}$ are known as *equipotential curves*, while the curves $v(x, y)$ are known as *streamlines*. (Each particle in a fluid of the type stated above moves along a streamline.)

The velocity $\mathbf{V}$ of fluid flow given by the function $f(z)$ at every point $z = x + iy$ is defined, in both magnitude

and direction, by the complex number

$$\mathbf{V} = Ve^{i\varphi} = V_{Ox} + iV_{Oy} = \overline{f'(z)} = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y},$$

which is the complex conjugate of the derivative of the complex potential at point z. The velocity's magnitude is

$$V = |\mathbf{V}| = |\overline{f'(z)}| = |f'(z)|,$$

while its direction forms an angle

$$\varphi = \arg \overline{f'(z)} = -\arg f'(z)$$

with the positive direction of the x axis. The projections V_{Ox} and V_{Oy} of the velocity vector $\mathbf{V}$ on the x and y axes are

$$V_{Ox} = \mathrm{pr}_{Ox}\mathbf{V} = \frac{\partial u}{\partial x}, \quad V_{Oy} = \mathrm{pr}_{Oy}\mathbf{V} = \frac{\partial u}{\partial y},$$

or, in view of the Cauchy-Riemann conditions,

$$V_{Ox} = \frac{\partial v}{\partial y}, \quad V_{Oy} = -\frac{\partial v}{\partial x}.$$

This means that

$$\mathbf{V} = \operatorname{grad} u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}.$$

We will assume that the coordinates $V_{Ox} = V_{Ox}(x, y)$ and $V_{Oy} = V_{Oy}(x, y)$ of the velocity vector $\mathbf{V}$ are continuously differentiable functions within their domains except, perhaps, at a finite number of points.

Definition 1. The quantity

$$\Gamma_L = \oint_L V_{Ox}\,dx + V_{Oy}\,dy = \oint_L \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$= \oint_L \frac{\partial u}{\partial x}\,dx - \frac{\partial v}{\partial x}\,dy$$

is called the *circulation of the velocity vector* $\mathbf{V}$ *along the closed curve* L traversed in the positive sense. The circulation Γ_L characterizes the "vorticity" of the fluid flow. We also assume that L does not pass through any singular points of V while there is only a finite number of such points in the interior of L.

Definition 2. The quantity

$$N_L = \oint_L V_{Ox}\,dy - V_{Oy}\,dx = \oint_L \frac{\partial u}{\partial x}\,dy - \frac{\partial u}{\partial y}\,dx$$

$$= \oint_L \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy$$

is called the *flux of vector through the closed curve L*. The curve L is traversed in the positive sense, i.e. the domain bounded by L remains to the left (we take the outward normal to L). We also assume that L does not pass through any singular points of V, or $f'(z)$. The flux N_L of the velocity vector V gives the amount of fluid passing through L per unit time.

The formulas defining the circulation Γ_L and the flux N_L can be combined thus:

$$\Gamma_L + iN_L = \oint_L f'(z)\,dz, \tag{13.2}$$

which enables finding the circulation and the flux residue calculations. When $f'(z)$ (and hence $\mathbf{V} = \overline{f'(z)}$) has a finite number of singular points z_k $(k = 1, 2, \ldots, n)$, we have

$$\Gamma_L + iN_L = 2\pi i \sum_{k=1}^{n} \operatorname{Res} f'(z_k).$$

The points where $\mathbf{V} = 0$, and hence $f'(z) = 0$, are called the *critical points of the flow*.

Example 1. Fluid motion is given by the complex potential $f(z) = z^2$. Find the velocity potential, the stream function, the equipotential curves, the streamlines, the magnitude and direction of the velocity vector V, and the projections V_{Ox} and V_{Oy} of V on the x and y axes.

Solution. Putting $z = x + iy$, we find that

$$f(z) = (x^2 - y^2) + i2xy,$$

which means that the velocity potential is $u(x, y) = x^2 - y^2$, the stream function is $v(x, y) = 2xy$, the equipotential lines $u(x, y) = \text{const}$ are the hyperbolas $x^2 - y^2 = \text{const}$, and the streamlines $v(x, y) = \text{const}$ are the hyperbolas $xy = \text{const}$. The magnitude of the velocity vector is

$$V = |\overline{f'(z)}| = |\overline{2z}| = 2\sqrt{x^2 + y^2}.$$

The direction of V is given by the angle

$$\varphi = -\arg f'(z) = -\arctan(y/x).$$

The projections of V on the x and y axes are

$$V_{Ox} = \frac{\partial u}{\partial x} = 2x, \quad V_{Oy} = \frac{\partial u}{\partial y} = -2y.$$

The origin of coordinates O (0, 0) is the critical (rest) point in the fluid flow.

Example 2. Fluid motion is given by the complex potential $f(z) = \ln \sinh \pi z$. Find the flux N_L through the circle $2|z| = 3$ and the circulation Γ_L along the same circle.

Solution. We start by finding the derivative of the complex potential:

$$f'(z) = \pi \coth \pi z.$$

Employing (13.2), we find that

$$\Gamma_L + iN_L = \pi \int_{|z|=3/2} \coth \pi z \, dz = \pi \int_{|z|=3/2} \frac{\cosh \pi z}{\sinh \pi z} \, dz.$$

The integrand has three simple poles inside the circle $2|z| = 3$, namely, $z_1 = -i$, $z_2 = 0$, and $z_3 = i$. Its residues at these poles are

$$\operatorname{Res} f'(z_k) = \pi \frac{1}{\pi} = 1, \quad k = 1, 2, 3.$$

Then

$$\int_{|z|=3/2} \coth \pi z \, dz = 6\pi i.$$

Consequently,

$$\Gamma_L + iN_L = 6\pi^2 i.$$

This means that the circulation is zero and the flux is $6\pi^2$.

Example 3. Find the complex potential $f(z)$ of fluid flow if the equipotential curves are given by the equation:

$$\cosh x \sin y + 2xy = c,$$

where $c = \text{const}$, and $f(0) = 0$.

Solution. The above condition implies that the velocity potential $u(x, y)$, i.e. the real part of the analytic function $f(z)$, which is what we wish to find, is

$$u = 2xy + \cosh x \sin y.$$

The problem is reduced, therefore, to constructing an analytic function from its real part. Specifically, we wish to find $f(z)$ from its real part $u(x, y)$. We have

$$\frac{\partial u}{\partial x} = 2y + \sinh x \sin y.$$

The first Cauchy-Riemann condition $\partial u/\partial x = \partial v/\partial y$ implies that

$$\frac{\partial v}{\partial y} = 2y + \sinh x \sin y.$$

From this we find that

$$v(x, y) = y^2 - \sinh x \cos y + \varphi(x),$$

where $\varphi(x)$ is an arbitrary differentiable function. Finding the derivative of $v(x, y)$ with respect to x and using the second Cauchy-Riemann condition, we have

$$\frac{\partial v}{\partial x} = -\cosh x \cos y + \varphi'(x) = -\frac{\partial u}{\partial y} = -2x - \cosh x \cos y,$$

whence $\varphi'(x) = -2x$, which means that $\varphi(x) = -x^2 + c$, with c a constant. Thus,

$$v(x, y) = y^2 - x^2 - \sinh x \cos y + c.$$

Consequently,

$$f(z) = 2xy + \cosh x \sin y + i(y^2 - x^2 - \sinh x \cos y) + ic.$$

From the fact that $f(0) = 0$ we find that $0 + i0 = c$, or $c = 0$. Thus, the sought complex potential is

$$f(z) = 2xy + \cosh x \sin y + i(y^2 - x^2 - \sinh x \cos y),$$

or

$$f(z) = -i(z^2 + \sinh z).$$

Fluid flow is defined by the complex potential. Find the velocity potential, the stream function, the equipotential curves, the streamlines, the magnitude and direction of the velocity vector, and the velocity projections on the coordinate axes.

505. $f(z) = z^2 + 2z + 2$.

506. $f(z) = 1/z^2$.

507. $f(z) = \ln(z - 1)$.

508. Construct the complex potential of fluid flow if the equipotential curves are given by the equation

$$x^2 - y^2 + 2xy + x = \text{const} \quad \text{and} \quad f(0) = 0.$$

509. Construct the complex potential of fluid flow if the streamlines are given by the equation

$$\cos x \sinh y = \text{const} \quad \text{and} \quad f(0) = 0.$$

510. Find the circulation along the circle $|z \pm a| = a$ if the complex potential of fluid flow is

$$f(z) = 5i \ln(z^2 - a^2).$$

Chapter 2

Operational Calculus

§ 14 Finding Object and Result Functions

An *object function* is any complex-valued function $f(t)$ of a real argument t that satisfies the following three conditions:

1°. $f(t)$ is integrable in any finite segment of the t axis (i.e. locally integrable).

2°. $f(t) = 0$ for all negative values of t.

3°. $|f(t)|$ grows as $t \to +\infty$ no faster than the exponential function, i.e. there are constants $M > 0$ and s such that for all values of t,

$$|f(t)| \leqslant Me^{st}. \tag{14.1}$$

The greatest lower bound s_0 of all numbers s for which (14.1) is valid is known as the *growth index of* $f(t)$.

Example 1. Show that the function

$$f(t) = \begin{cases} e^{2t} \sin 3t, & t > 0, \\ 0, & t < 0, \end{cases}$$

is an object function.

Solution. Indeed, $f(t)$ is locally integrable, since

$$\int_{t_1}^{t_2} e^{2t} \sin 3t \, dt$$

exists for all finite t_1 and t_2. Condition 2° is met by the nature of the function. Finally, for all t real,

$$|e^{2t} \sin 3t| \leqslant e^{2t},$$

which means that for M in condition 3° we can take any number that is no less than unity, and $s_0 = 2$.

A simple object function is the unit step function

$$\eta(t)=\begin{cases}1, & t>0,\\ 0, & t<0.\end{cases}$$

Obviously,

$$\varphi(t)\,\eta(t)=\begin{cases}\varphi(t), & t>0,\\ 0, & t<0,\end{cases}$$

so that if $\varphi(t)$ satisfies 1° and 3°, $\varphi(t)\,\eta(t)$ satisfies all the conditions imposed on object functions.

511. Check whether the functions below can serve as object functions:

(a) $f(t)=b^t\eta(t)$, $b>0$, $b\neq 1$;

(b) $f(t)=e^{(2+4i)t}\eta(t)$; (c) $f(t)=\dfrac{1}{t-3}\eta(t)$;

(d) $f(t)=t^2\eta(t)$; (e) $f(t)=\cosh(3-i)\,t\eta(t)$;

(f) $f(t)=\tan t\eta(t)$; (g) $f(t)=t^t\eta(t)$;

(h) $f(t)=e^{-t}\cos t\eta(t)$; (i) $f(t)=e^{t^2}\eta(t)$;

(j) $f(t)=e^{-t^2}\eta(t)$; (k) $f(t)=\dfrac{1}{t^2+2}\eta(t)$;

(l) $f(t)=\eta(t)+\sum\limits_{k=1}^{\infty}(-1)^k\eta(t-k)$.

In what follows we will as a rule write $f(t)$ instead of $f(t)\,\eta(t)$, assuming that the functions we are considering are continued by zero for negative values of t.

A *result function* $F(p)$ of the complex variable $p=s+i\sigma$ is associated with its object function $f(t)$ through the (*one-sided*) *Laplace transformation*:

$$F(p)=\int_0^{+\infty}f(t)\,e^{-pt}\,dt. \qquad (14.2)$$

The fact that $F(p)$ is the result function of $f(t)$ is usually written thus:

$$f(t)\doteqdot F(p).$$

$F(p)$ is defined in the half-plane $\operatorname{Re} p=s>s_0$ and is analytic there.

Example 2. Using the definition of result function, find the result function of

$$f(t)=e^{2t}.$$

Solution. We see that in our case $s_0 = 2$. For this reason the result function $F(p)$ will at least be defined and analytic in the half-plane $\operatorname{Re} p > 2$. We have

$$F(p) = \int_0^{+\infty} e^{2t} e^{-pt}\, dt = \int_0^{+\infty} e^{-(p-2)t}\, dt$$

$$= \frac{1}{-(p-2)} e^{-(p-2)t} \Big|_{t=0}^{t=+\infty} = \frac{1}{p-2} \quad (\operatorname{Re} p = s > 2).$$

Thus $F(p) = (p-2)^{-1}$. This function is analytic when $\operatorname{Re} p$ is greater than two and, besides, is analytic everywhere except at point $p = 2$. This does not contradict the above statement concerning the analyticity of $F(p)$, since the latter guarantees only that $F(p)$ is analytic when $\operatorname{Re} p$ is greater than s_0 but does not state that when $\operatorname{Re} p$ is less than s_0, $F(p)$ is nowhere analytic.

Using the definition of a result function find the result functions of the following functions:

512. $f(t) = t$. 513. $f(t) = \sin 3t$.

514. $f(t) = te^t$. 515. $f(t) = t^\alpha$ $(\alpha > -1)$.

516. Can $\varphi(p) = (\cos p)^{-1}$ be the result function of an object function?

Properties of the Laplace transformation

The uniqueness theorem *The Laplace transform*

$$F(p) = \int_0^{+\infty} e^{-p} f(t)\, dt$$

is unique in the sense that two functions $f_1(t)$ and $f_2(t)$ with the same Laplace transforms coincide at all points where they are continuous when $t > 0$.

Different discontinuous functions may nevertheless have the same Laplace transforms (the reader is advised to construct such functions).

I. *Linearity.* For any complex constants α and β,

$$\alpha f(t) + \beta g(t) \doteqdot \alpha F(p) + \beta G(p)$$

(here and everywhere below we assume that $f(t) \doteqdot F(p)$ and $g(t) \doteqdot G(p)$).

Find the result functions of the following functions:

517. $1 + t$. 518. $2 \sin t - \cos t$.

519. $t + (1/2)\, e^{-t}$.

II. *Change of scale.* For any positive α,

$$f(\alpha t) \doteqdot \frac{1}{\alpha} F\left(\frac{p}{\alpha}\right).$$

Using the change-of-scale property, find the result functions of the following functions:

520. $f(t) = e^{at}$. 521. $f(t) = \sin 4t$.

522. (a) $f(t) = \cos \omega t$; (b) $f(t) = \sinh 3t$.

523. Let $f(t) \doteqdot F(p)$. Find the result function of $f(t/a)$, with $a > 0$, directly and by applying the change-of-scale property.

Using the properties of linearity and change of scale, find the result functions of the following functions:

524. $f(t) = \sin^2 t$. 525. $f(t) = \sin mt \cos nt$.

526. $f(t) = \cos^3 t$. 527. $f(t) = \sin mt \sin nt$.

528. $f(t) = \sin^4 t$. 529. $f(t) = \cos mt \cos nt$.

III. *Differentiation of object function.* If $f(t)$, $f'(t)$, ... , $f^{(n)}(t)$ are object functions and $f(t) \doteqdot F(p)$, then

$$f'(t) \doteqdot pF(p) - f(0),$$

$$f''(t) \doteqdot p^2F(p) - pf(0) - f'(0),$$

. . . . . .

$$f^{(n)}(t) \doteqdot p^nF(p) - p^{n-1}f(0) - p^{n-2}f'(0) - \ldots - f^{(n-1)}(0),$$

where $f^{(k)}(0)$ $(k = 1, 2, \ldots, n-1)$ is understood as the limit of $f^{(k)}(t)$ as $t \to +0$.

Example 3. Using the property of differentiation of object function, find the result function of $f(t) = \sin^2 t$.

Solution. Let $f(t) \doteqdot F(p)$. Then

$$f'(t) \doteqdot pF(p) - f(0).$$

But $f(0) = 0$ and $f'(t) = 2 \sin t \cos t = \sin 2t \doteqdot 2/(p^2 + 4)$. Consequently, $2/(p^2 + 4) = pF(p)$, which implies that

$$F(p) = \frac{2}{p(p^2+4)} \doteqdot \sin^2 t.$$

Using the property of differentiation of object function, find the result functions of the following functions:

530. $f(t) = \cos^2 t$. 531. $f(t) = \sin^3 t$.

532. $f(t) = t \sin \omega t$. 533. $f(t) = \cos^4 t$.

534. $f(t) = t \cos \omega t$. 535. $f(t) = te^t$.

IV. *Differentiation of result function.* This is done by multiplying the corresponding object function by $-t$:

$$F'(p) \doteqdot -tf(t),$$

or, in general,

$$F^{(n)}(p) \doteqdot (-t)^n f(t).$$

Example 4. Find the result function of

$$f(t) = t^2 e^t.$$

Solution. We have $e^t \doteqdot (p-1)^{-1}$. By the property of differentiation of result function, $\left(\frac{1}{p-1}\right)' \doteqdot -te^t$, whence $\frac{1}{(p-1)^2} \doteqdot te^t$. Moreover,

$$\left[\frac{1}{(p-1)^2}\right]' \doteqdot -t(te^t), \quad \text{or} \quad \frac{2!}{(p-1)^3} \doteqdot t^2 e^t.$$

Find the result functions of the following functions:

536. $f(t) = t^2 \cos t$. 537. $f(t) = t(e^t + \cosh t)$.

538. $f(t) = (t+1) \sin 2t$. 539. $f(t) = t \sinh 3t$.

V. *Integration of object function.* This is done by dividing the corresponding result function by p, i.e. if $f(t) \doteqdot F(p)$, then

$$\int_0^t f(\tau)\, d\tau \doteqdot \frac{F(p)}{p}.$$

Example 5. Find the result function of $\int_0^t e^\tau d\tau$.

Solution. We have $e^t \doteqdot (p-1)^{-1}$. By the property of integration of object function,

$$\int_0^t e^\tau\, d\tau \doteqdot \frac{\frac{1}{p-1}}{p} = \frac{1}{p(p-1)}.$$

Find the result functions of the following functions:

540. $f(t) = \int_0^t \sin \tau \, d\tau$.

541. $f(t) = \int_0^t (\tau + 1) \cos \omega\tau \, d\tau$.

542. $f(t) = \int_0^t \tau \sinh 2\tau \, d\tau$.

543. $f(t) = \int_0^t \cos^2 \omega\tau \, d\tau$.

544. $f(t) = \int_0^t \cosh \omega\tau \, d\tau$.

545. $f(t) = \int_0^t \tau^2 e^{-\tau} \, d\tau$.

VI. *Integration of result function.* If $\int_p^\infty F(p)\,dp$ is convergent, it is the result function of $f(t)/t$:

$$\frac{f(t)}{t} \doteq \int_p^\infty F(p)\,dp.$$

Example 6. Find the result function of $(\sin t)/t$.

Solution. We know that $\sin t \doteq (p^2 + 1)^{-1}$, which means that

$$\frac{\sin t}{t} \doteq \int_p^\infty \frac{dp}{p^2+1} = \arctan p \,|_p^\infty = \frac{\pi}{2} - \arctan p = \operatorname{arc\,cot} p$$

(for multiple-valued function such as Ln z, Arc tan z we take their principal branches, for which ln 1 = 0, arc tan 1 = $= \pi/4$, etc.).

Find the result functions of the following functions:

546. (a) $\frac{e^t - 1}{t}$; (b) $\frac{1 - e^{-t}}{t}$; (c) $\frac{\sin^2 t}{t}$.

547. (a) $\frac{1-\cos t}{t}$; (b) $\frac{\cos t-\cos 2t}{t}$.

548. (a) $\frac{e^t-1-t}{t}$; (b) $\frac{e^t-e^{-t}}{t}$.

The property of integration of result function can be used to evaluate some improper integrals. Suppose $f(t) \doteqdot f(p)$ and let the improper integral of $f(t)/t$ from 0 to ∞ be convergent. Then

$$\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty F(p)\,dp, \tag{14.3}$$

where the integral on the right-hand side can be evaluated along the real positive semiaxis.

Example 7. Evaluate the integral $\int_0^\infty \frac{\sin t}{t}\,dt$.

Solution. We know that $\sin t \doteqdot \frac{1}{p^2+1}$. By (14.3) we have

$$\int_0^\infty \frac{\sin t}{t}\,dt = \int_0^\infty \frac{dz}{z^2+1} = \arctan z\,|_0^\infty = \frac{\pi}{2}.$$

Evaluate the following integrals:

549. $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt \quad (a>0,\ b>0)$.

550. $\int_0^\infty \frac{e^{-\alpha t}\sin at}{t}\,dt \quad (\alpha>0,\ a>0)$.

551. $\int_0^\infty \frac{e^{-\alpha t}-e^{-\beta t}}{t}\sin mt\,dt \quad (\alpha>0,\ \beta>0,\ m>0)$.

552. $\int_0^\infty \frac{Ae^{-\alpha t}+Be^{-\beta t}+Ce^{-\gamma t}+De^{-\delta t}}{t}\,dt$ $(A+B+C+D=0,$ $\alpha>0,\ \beta>0,\ \gamma>0,\ \delta>0)$.

553. $\int_0^\infty \frac{\cos at-\cos bt}{t}\,dt \quad (a>0,\ b>0)$.

554. $\int_0^\infty \frac{\sin at \sin bt}{t}\,dt \quad (a>0,\ b>0)$.

VII. *Translation of result function.* If $f(t) \doteqdot F(p)$, then for any complex-valued p_0 we have[1]

$$e^{p_0 t} f(t) \doteqdot F(p - p_0). \tag{14.4}$$

Example 8. Find the result function of $f(t) = e^{-t}\cos 2t$.

Solution. We know that $\cos 2t \doteqdot p/(p^2+4)$. By the property of translation of result function ($p_0 = -1$) we have

$$e^{-t}\cos 2t \doteqdot \frac{p+1}{(p+1)^2+4}.$$

Find the result functions of the following functions:

555. (a) $e^{2t}\sin t$; (b) $e^t \cos nt$. 556. $e^{-t}t^3$.

557. $e^t \sinh t$. 558. $te^t \cos t$. 559. $e^{3t}\sin^2 t$.

560. $e^{-\alpha t}\cos^2 \beta t$.

VIII. *Translation of object function.* If $f(t) \doteqdot F(p)$, then for any positive τ we have

$$f(t-\tau) \doteqdot e^{-p\tau} F(p). \tag{14.5}$$

This property is often used when one must find the result function of a function that in different segments is given by different analytic expressions.

Example 9. Find the result function of

$$f(t-1) = (t-1)^2 \eta(t-1).$$

Solution. For $f(t) = t^2\eta(t)$ we have

$$f(t) \doteqdot \frac{2}{p^3}.$$

By the property of translation of the object function $(t-1)^2\eta(t-1)$ we have

$$(t-1)^2\eta(t-1) \doteqdot e^{-p}\frac{2}{p^3}.$$

It is essential that we are looking for the result function of $(t-1)^2\eta(t-1)$, the latter being zero when $t<1$. If we were to take $f_1(t) = (t-1)^2\eta(t)$, we would have $f_1(t) = (t^2 - 2t + 1)\eta(t)$, which by the property of linearity yields

$$(t-1)^2\eta(t) \doteqdot \frac{2}{p^3} - \frac{2}{p^2} + \frac{1}{p}.$$

Find the result functions of the following functions:

561. $\sin(t-b)\,\eta(t-b)$,

562. $\cos^2(t-b)\,\eta(t-b)$.

563. $e^{t-2}\eta(t-2)$.

Example 10. Find the result function $F(p)$ of a function $f(t)$ given in the form of the following graph (Fig. 39):

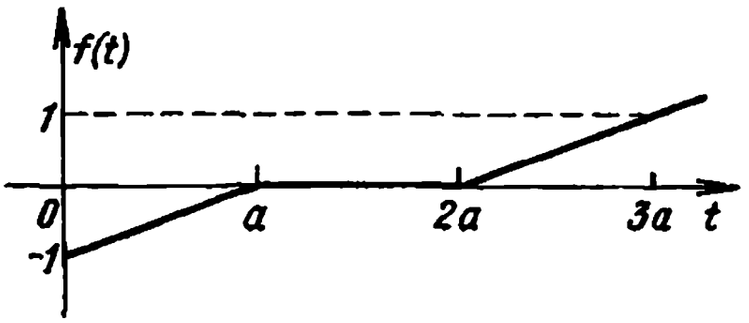


Figure 39

Solution. Let us find the analytic expression for $f(t)$.

(a) For $t \in (0, a)$ we have

$$f(t) = \frac{t-a}{a}\,\eta(t). \tag{14.6}$$

(b) For $t \in (a, 2a)$ we have $f(t) = 0$.

(c) For $t \geqslant 2a$ we have

$$f(t) = \frac{t-2a}{a}\,\eta(t-2a). \tag{14.7}$$

Assuming that $f(t)$ is given by (14.6) for all $t \geqslant 0$, let us find the function $\psi_1(t)$ that we must add to $f(t)$ to obtain a function $f(t) = 0$ for all $t \geqslant a$. If we require that

$$\frac{t-a}{a} + \psi_1(t) = 0,$$

for $t \geqslant a$, then we will find that

$$\psi_1(t) = -\frac{t-a}{a}\,\eta(t-a).$$

Next we find a function $\psi_2(t)$ which when added to $f(t) \equiv 0$ yields $(t-2a)/a$ for all $t \geqslant 2a$. This yields

$$0 + \psi_2(t) = \frac{t-2a}{a},$$

whence

$$\psi_2(t) = \frac{t-2a}{a}\,\eta(t-2a).$$

Thus, for all $t \geqslant 0$ we have

$$f(t) = \frac{t-a}{a}\eta(t) - \frac{t-a}{a}\eta(t-a) + \frac{t-2a}{a}\eta(t-2a).$$

Using the properties of linearity and translation of object function, we can find the sought result function $F(p)$ of the given function $f(t)$:

$$F(p) = \frac{1}{ap^2} - \frac{1}{p} - \frac{1}{ap^2}e^{-ap} - \frac{1}{ap^2}e^{-2ap}.$$

Example 11. Find the result function $F(p)$ of a function $f(t)$ given in the form of the following graph (Fig. 40):

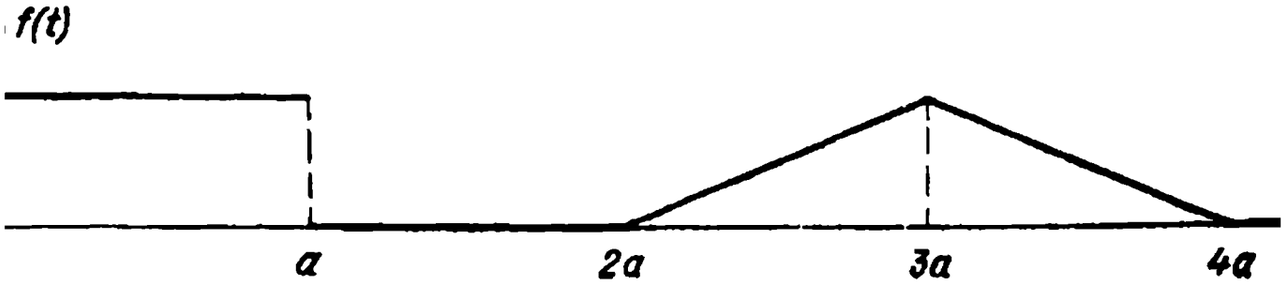


Figure 40

Solution. Let us find the analytic expression for $f(t)$.

(a) $f(t) = 1$ for $t \in (0, a)$,
(b) $f(t) = 0$ for $t \in (a, 2a)$.
(c) $f(t) = (t - 2a)/a$ for $t \in (2a, 3a)$,
(d) $f(t) = 1 - (t - 3a)/a$ for $t \in (3a, 4a)$,
(e) $f(t) = 0$ for $t \geqslant 4a$.

For $t \in (0, a)$ we have $f(t) = 1$.

Next we find a function $\psi_1(t)$ which for $t \geqslant a$ satisfies the following relationship: $1 + \psi_1(t) = 0$, which means that $\psi_1(t) = -1 \times \eta(t - a)$.

Now we find a function $\psi_2(t)$ which for $t > 2a$ satisfies the following relationship: $0 + \psi_2(t) = (t - 2a)/a$. Whence

$$\psi_2(t) = \frac{t-2a}{a}\eta(t-2a).$$

In a similar manner we can find the functions

$$\psi_3(t) = -2\frac{t-3a}{a}\eta(t-3a), \quad \psi_4(t) = \frac{t-4a}{a}(t-4a).$$

Thus,

$$f(t)=\eta(t)-\eta(t-a)+\frac{t-2a}{a}\eta(t-2a)$$
$$-2\frac{t-3a}{a}\eta(t-3a)+\frac{t-4a}{a}\eta(t-4a).$$

Using the properties of linearity and translation of object function, we can find the sought result function:

$$F(p)=\frac{1}{p}-\frac{1}{p}e^{-ap}+\frac{1}{ap^2}e^{-2ap}-\frac{2}{ap^2}e^{-3ap}+\frac{1}{ap^2}e^{-4ap}.$$

Example 12. Find the result function of

$$f(t)=\begin{cases}0 \text{ for } t<1,\\ t^2 \text{ for } 1<t<2,\\ 0 \text{ for } t>2.\end{cases}$$

Solution. We express $f(t)$ in terms of powers of $t-1$ and $t-2$. We have

$$t^2=[(t-1)+1]^2=(t-1)^2+2(t-1)+1,$$
$$t^2=[(t-2)+2]^2=(t-2)^2+4(t-2)+4.$$

Therefore, we can write

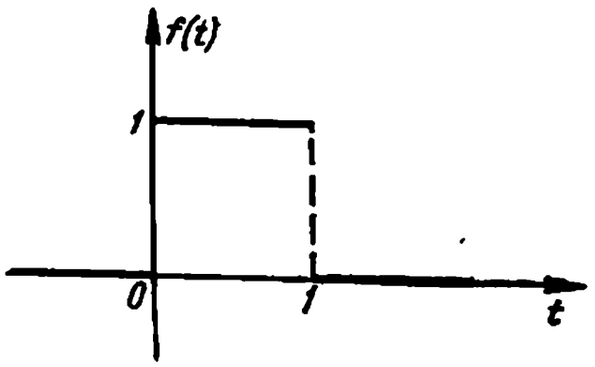
$$f(t)=[(t-1)^2+2(t-1)+1]\eta(t-1)$$
$$-[(t-2)^2+4(t-2)+4]\eta(t-2).$$

Going over to result functions, we obtain

$$f(t)\doteq F(p)=\left(\frac{2}{p^3}+\frac{2}{p^2}+\frac{1}{p}\right)e^{-p}-\left(\frac{2}{p^3}+\frac{4}{p^2}+\frac{4}{p}\right)e^{-2p}.$$

Find the result functions of the functions given below by graphs:

564.



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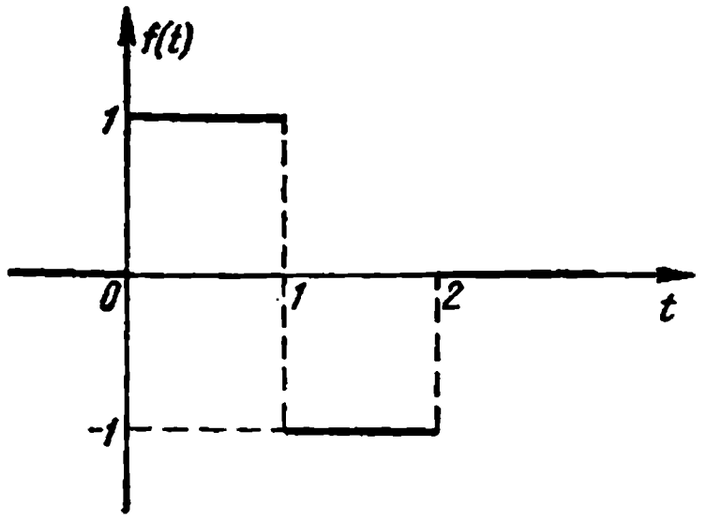
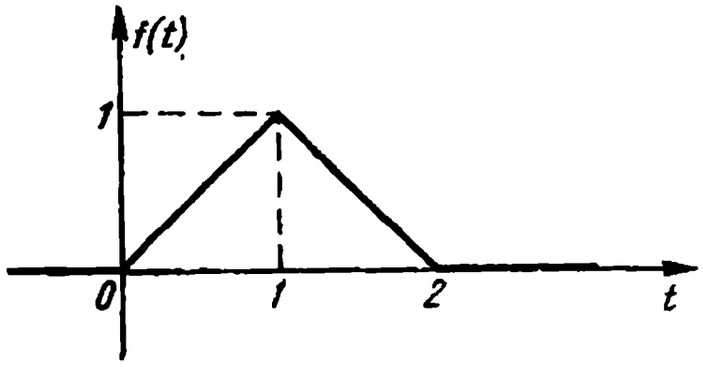


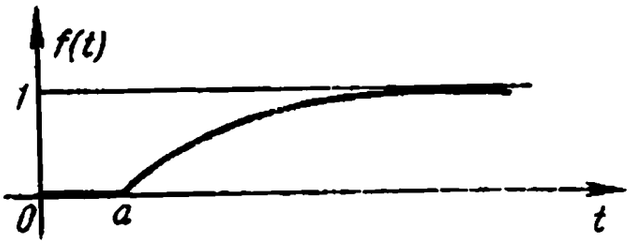
566.



567.

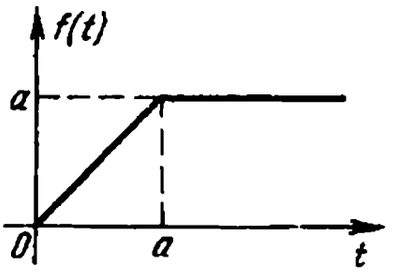
$$f(t)=\begin{cases}0 & \text{at } 0\leqslant t\leqslant a,\\ e^{-b(t-a)} & \text{at } t>a.\end{cases}$$

568.

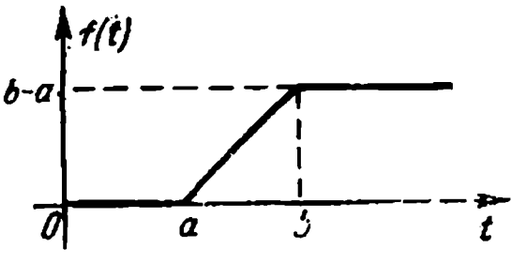


$$f(t)=\begin{cases}0 & \text{at } 0\leqslant t\leqslant a,\\ 1-e^{-b(t-a)} & \text{at } t>a.\end{cases}$$

569.



570.



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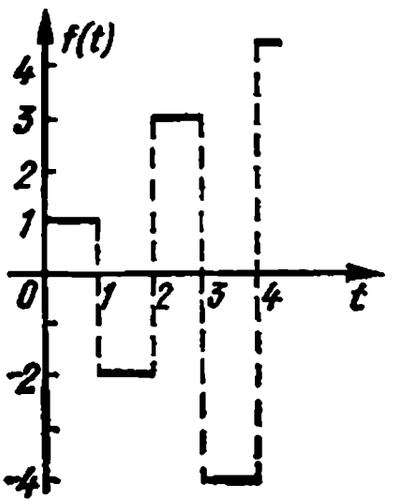

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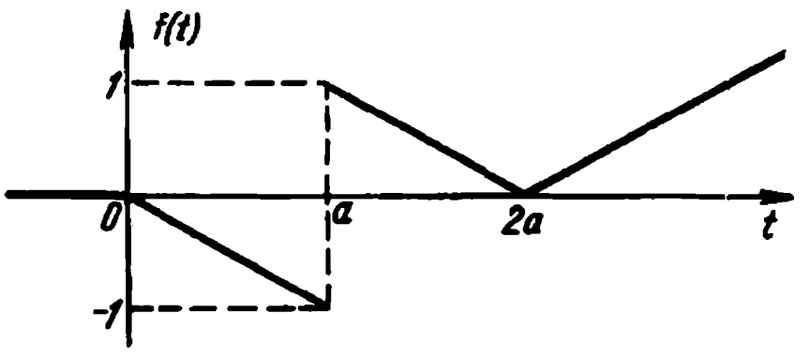

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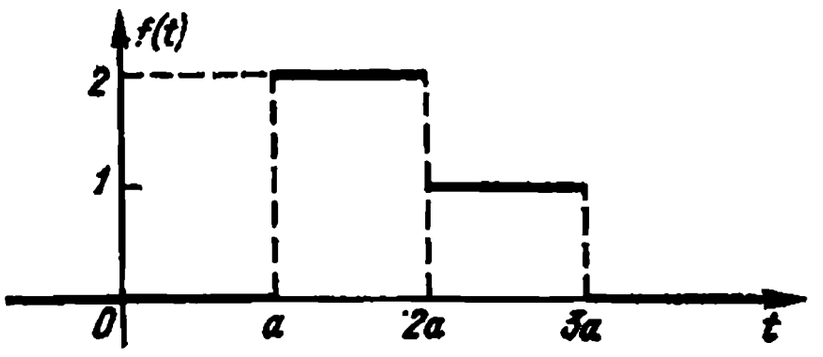

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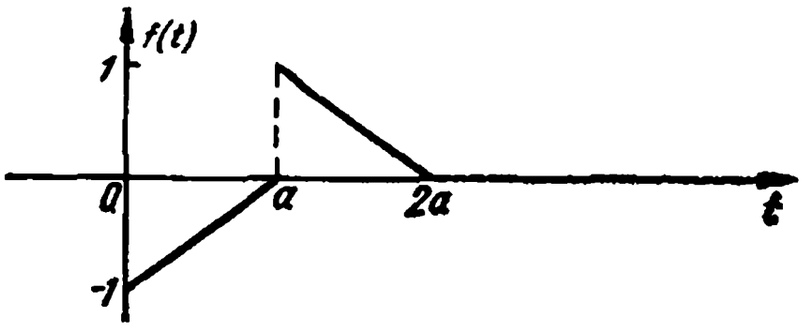

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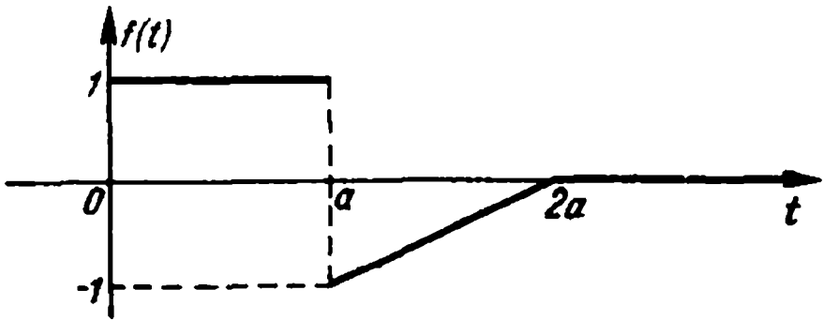

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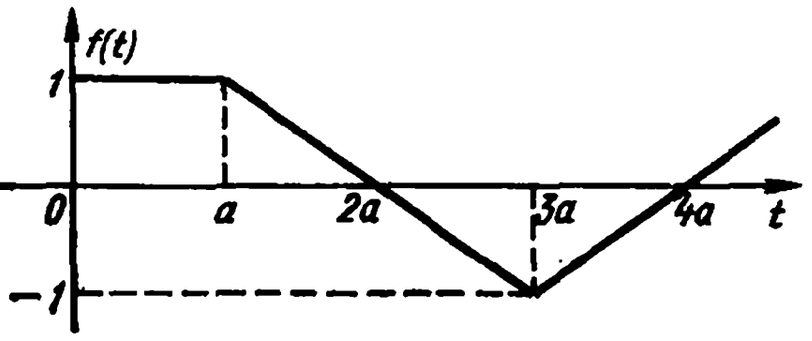

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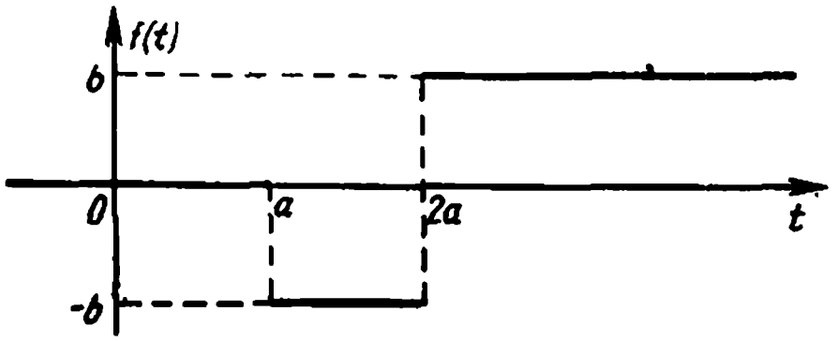


578. Su pose $f(t)$ is periodic with a period T and is an object function. Show that the corresponding result function (the Laplace transform) $F(p)$ is given by the formula

$$F(p)=\frac{1}{1-e^{-pT}}\int_0^T e^{-pT} f(t)\,dt$$

and is defined in the half-plane $\operatorname{Re} p = s > 0$.

Example 13. Find the result function of the periodic function $f(t)$ given by the graph in Fig. 41.

Solution. The result function can be found by the formula

$$F(p) = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)\, dt, \tag{14.8}$$

where $f(t)$ is the periodic function (period T), and $\operatorname{Re} p = s > 0$. Substituting into (14.8) the expression

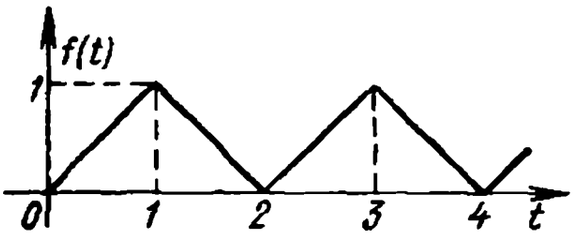


Figure 41

$$f(t) = \begin{cases} t & \text{at } 0 \leqslant t \leqslant 1, \\ 2-t & \text{at } 1 < t \leqslant 2, \end{cases}$$

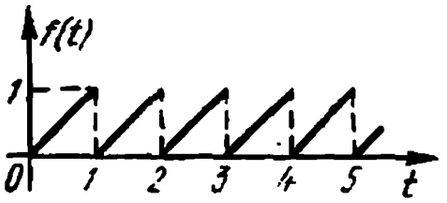
and bearing in mind that $T = 2$, we find that

$$F(p) = \frac{1}{1-e^{-2p}} \left[\int_0^1 te^{-pt}\, dt + \int_1^2 (2-t)\, e^{-pt}\, dt \right] = \frac{1-e^{-p}}{p^2(1+e^{-p})}.$$

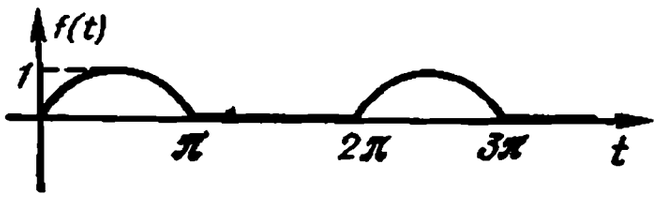
Find the result functions of the following periodic functions:

579.

580.



581. $f(t) = |\sin t|$. 582. $f(t) = |\cos t|$.

583. 

$$f(t)\begin{cases}\sin t & \text{for } 2k\pi < t < (2k+1)\pi,\\ 0 & \text{for } (2k+1)\pi < t < (2k+2)\pi\end{cases}\quad (k=0,\ 1,\ 2,\ \ldots).$$

584. Show that if $f(t) \doteq F(p)$, then

$$f(t)\,\eta(t-a) \doteq F(p) - \int_0^a f(t)\,e^{-pt}\,dt.$$

Operational calculus often uses the so-called generalized functions (or distributions), which play an important role in mathematical physics. (A rigorous definition of a generalized function can be found in I. M. Gel'fand *et al.*, *Generalized Functions*, Academic Press, New York, 1964.)

The *Dirac delta function* $\delta(t)$ is one example of a generalized function. It is defined thus:

$$(1)\quad \delta(t) = \begin{cases} 0 & \text{if } t \neq 0,\\ \infty & \text{if } t = 0,\end{cases} \qquad (2)\quad \int_\alpha^\beta \delta(t)\,f(t)\,dt = f(0),$$

where (α, β) is any interval with point $t = 0$, and $f(t)$ is a function continuous at $t = 0$. The function $\delta(t - \tau)$, concentrated at $t = \tau$, is defined similarly.

The theory of generalized functions considers $\delta(t)$ as the first derivative of the unit step function $\eta(t) = \begin{cases} 1, & t > 0,\\ 0, & t < 0,\end{cases}$ i.e.

$$\eta'(t) = \delta(t). \tag{14.9}$$

Similarly, for any τ,

$$\eta'(t-\tau) = \delta(t-\tau).$$

Note that the derivative of $\eta(t)$ in the conventional sense is zero for all $t \neq 0$ and does not exist for $t = 0$.

The following formulas are true:

$$\left.\begin{aligned} \delta(t) &\doteq 1,\\ \delta^{(m)}(t) &\doteq p^m \quad (m \text{ a nonnegative integer}),\\ \delta(t-\tau) &\doteq e^{-p\tau}\end{aligned}\right\} \tag{14.10}$$

Consider a function $f(t)$ that has ordinary discontinuities at points t_k $(k = 1, 2, \ldots, n)$ with jumps

$$h_k = f(t_k + 0) - f(t_k - 0) \quad (k = 1, 2, \ldots, n).$$

Suppose $f(t)$ is continuously differentiable in the intervals (t_k, t_{k+1}) $(k = 1, 2, \ldots, n-1)$ and for $t < t_1$ and $t > t_n$. Then

$$f'(t) = f_1'(t) + \sum_{k=1}^{n} h_k \delta(t - t_k), \qquad (14.11)$$

where $f_1(t) = f(t) - \sum_{k=1}^{n} h_k \eta(t - t_k)$ is the "joined" function. Thus, the derivative of a discontinuous function $f(t)$ is the sum of the conventional derivative $f_1'(t)$ (in the intervals where $f(t)$ is smooth) and the Dirac delta functions at the points of discontinuity with the jumps as coefficients. This rule is important for correct use of the theorem of operational calculus when dealing with discontinuous functions.

Let us take the following function $f(t)$ as an example:

$$f(t) = \eta(t) - 2\eta(t - 1) + \eta(t - 2).$$

Applying (14.11), we find that

$$f'(t) = \delta(t) - 2\delta(t - 1) + \delta(t - 2),$$

which, according to (14.10), yields

$$f'(t) \doteqdot 1 - 2e^{-p} + e^{-2p}.$$

Moreover,

$$f(t) \doteqdot \frac{1}{p} - \frac{2}{p} e^{-p} + \frac{1}{p} e^{-2p},$$

which yields

$$f(t) = \eta(t) - 2\eta(t - 1) + \eta(t - 2).$$

If we were to ignore (14.11), we would arrive at the following result. The derivative of $f(t)$ in the conventional sense is zero everywhere except at points $t = 0$, $t = 1$, and $t = 2$, where it does not exist. But then the Laplace transform of

$f'(t)$ would be zero, which means that the result function of $f(t)$ is zero, too, which, of course, is untrue.

585. Solve Problem 580 first finding the result function of the derivative of $f(t)$ and then the result function of $f(t)$.

586. Let a and b be two positive numbers and $f(t) \doteqdot F(p)$. Show that the result function of

$$g(t) = \begin{cases} f(at-b), & t > \frac{b}{a}, \\ 0, & t < \frac{b}{a}, \end{cases}$$

is $\frac{1}{a} e^{-\frac{b}{a}p} F\left(\frac{p}{a}\right)$ (change of scale and translation of object function).

587. Find the result functions of the following functions

(a) $f(t) = \begin{cases} \sin\left(2t - \frac{\pi}{4}\right), & t > \frac{\pi}{8}, \\ 0, & t < \frac{\pi}{8}; \end{cases}$

(b) $f(t) = \begin{cases} \cos\left(3t - \frac{\pi}{6}\right), & t > \frac{\pi}{18}, \\ 0, & t > \frac{\pi}{18}; \end{cases}$

(c) $f(t) = \begin{cases} \sinh(3t-6), & t > 2, \\ 0, & t < 2. \end{cases}$

588. Find the result function of the distribution function for masses m_k at points $t = k$:

$$f(t) = \sum_{k=0}^{n} m_k \delta(t-k).$$

IX. *Convolution of object functions.* The product of two result functions $F(p)$ and $\Phi(p)$ is also a result function and

$$F(p)\,\Phi(p) \doteqdot \int_0^t f(\tau)\,\varphi(t-\tau)\,d\tau. \qquad (14.12)$$

The integral on the right-hand side of (14.12) is known as the *convolution of* $f(t)$ and $\varphi(t)$ and is denoted by $f(t) * \varphi(t)$.

Example 14. Find the result function of

$$\psi(t) = \int_0^t (t-\tau)\, e^{\tau}\, d\tau.$$

Solution. The function $\psi(t)$ is the convolution of $f(t) = t$ and $\varphi(t) = e^t$. By the property of the convolution of object functions we have

$$\psi(t) \doteqdot \Psi(p) = F(p)\,\Phi(p) = \frac{1}{p^2}\,\frac{1}{p-1} = \frac{1}{p^2(p-1)}.$$

Example 15. Let $F_1(p) = 1/p^x$ and $F_2(p) = 1/p^y$ (with x and y real and positive). Then

$$f_1(t) = \frac{t^{x-1}}{\Gamma(x)}, \quad f_2(t) = \frac{t^{y-1}}{\Gamma(y)}.$$

By the property of convolution of object functions we have

$$F_1(p)\,F_2(p) = \frac{1}{p^{x+y}} \doteqdot \frac{1}{\Gamma(x)\,\Gamma(y)} \int_0^t (t-\tau)^{y-1}\,\tau^{x-1}\,d\tau. \quad (14.13)$$

On the other hand,

$$F_1(p)\,F_2(p) = \frac{1}{p^{x+y}} \doteqdot \frac{t^{x+y-1}}{\Gamma(x+y)}. \quad (14.14)$$

Combining (14.13) and (14.14) yields

$$\frac{t^{x+y-1}}{\Gamma(x+y)} = \frac{1}{\Gamma(x)\,\Gamma(y)} \int_0^t (t-\tau)^{y-1}\,\tau^{x-1}\,d\tau.$$

Putting $\tau = \lambda t$, we find that the last relationship leads to

$$\frac{t^{x+y-1}}{\Gamma(x+y)} = \frac{t^{x+y-1}}{\Gamma(x)\,\Gamma(y)} \int_0^1 \lambda^{x-1}(1-\lambda)^{y-1}\,d\lambda.$$

The integral on the right-hand side is Euler's integral of the first kind $B(x, y)$ (also called the complete beta function). We have arrived at an important formula that connects the gamma function Γ and the beta function B:

$$B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

Repeated convolution. Suppose we have three object functions: $f_1(t)$, $f_2(t)$, and $f_3(t)$. Then

$$F_1(p)\,F_2(p) \doteqdot \int_0^t f_2(\tau_1)\,f_1(t-\tau_1)\,d\tau_1.$$

Next,

$$(F_1(p)\,F_2(p))\,F_3(p)$$

$$\doteqdot \int_0^t \left\{ \int_0^{t-\tau_2} f_2(\tau_1)\,f_1(t-\tau_1-\tau_2)\,d\tau_1 \right\} f_3(\tau_2)\,d\tau_2$$

$$= \int_0^t f_3(\tau_2)\,d\tau_2 \int_0^{t-\tau_2} f_2(\tau_1)\,f_1(t-\tau_1-\tau_2)\,d\tau_1.$$

Let us assume that

$$F_1(p) = \frac{1}{p} \doteqdot \eta(t).$$

Then

$$\frac{F_2(p)\,F_3(p)}{p} \doteqdot \int_0^t f_3(\tau_2)\,d\tau_2 \int_0^{t-\tau_2} f_2(\tau_1)\,(t-\tau_1-\tau_2)\,d\tau_1.$$

From the definition of $\eta(t)$ it then follows that actually the integration is over the domain $t > \tau_1 + \tau_2$, where $\eta(t-\tau_1-\tau_2) = 1$. Thus,

$$\frac{F_2(p)\,F_3(p)}{p} \doteqdot \iint\limits_{\tau_1+\tau_2<t} f_2(\tau_1)\,f_3(\tau_2)\,d\tau_1\,d\tau_2.$$

We will use this result to calculate the volume of an n-dimensional sphere. Let us introduce in the n-dimensional space a Cartesian system of coordinates $x_1, x_2, \ldots, x_n$. The volume $V_n(R)$ of a hypersphere of radius R is given by the following formula:

$$V_n(R) = \iint\limits_{x_1^2+x_2^2+\ldots+x_n^2\leqslant R^2} \ldots \int dx_1\,dx_2 \ldots dx_n.$$

In view of the symmetry of a sphere with respect to its center we have

$$V_n(R) = 2^n \int_0^{x_1^2+x_2^2+\ldots+x_n^2<R^2} dx_1 \int_0 dx_2 \ldots \int_0 dx_n.$$

We change variables thus: $x_k = \sqrt{\tau_k}$ $(k = 1, 2, \ldots, n)$. Then

$$V_n(R) = \int\limits_{\tau_1+\tau_2+\ldots+\tau_n<R^2} \int \ldots \int \frac{d\tau_1\, d\tau_2 \ldots d\tau_n}{\sqrt{\tau_1}\,\sqrt{\tau_2}\ldots\sqrt{\tau_n}}.$$

This is the repeated convolution of n identical functions

$$f_k(t) = \frac{\eta(t)}{\sqrt{t}} \quad (k = 1, 2, \ldots, n), \quad \text{where } t = R^2.$$

We know the result function of $f_k(t)$ from Problem 515:

$$f_k(t) \doteq \frac{\sqrt{\pi}}{\sqrt{p}}.$$

Whence

$$V_n(\sqrt{t})\,\frac{(\sqrt{\pi})^n}{p^{1+n/2}} = \frac{\pi^{n/2}}{p^{1+n/2}},$$

which implies that

$$V_n(\sqrt{t}) = \frac{\pi^{n/2}}{\Gamma\left(1+\frac{n}{2}\right)}\, t^{n/2} = \frac{\pi^{n/2}R^n}{\Gamma\left(1+\frac{n}{2}\right)} = V_n(R).$$

Thus,

$$V_n(R) = \frac{\pi^{n/2}R^n}{\Gamma\left(1+\frac{n}{2}\right)}.$$

In particular, for $n = 2$ we find that

$$V_2(R) = \frac{\pi R^2}{\Gamma(2)} = \pi R^2.$$

Find the result functions of the following functions:

589. $\int\limits_0^t e^{t-\tau}\sin\tau\, d\tau.$

590. $\int\limits_0^t \cos(t-\tau)\, e^{2\tau}\, d\tau.$

591. $\int\limits_0^t (t-\tau)^2 \cosh\tau\, d\tau.$

592. $\int\limits_0^t (t-\tau)^n f(\tau)\, d\tau.$

593. $\int\limits_0^t e^{2(\tau-t)}\tau^2\, d\tau.$

The first expansion theorem

If $F(p)$ is analytic in the neighborhood of the point at infinity and is zero at that point and if the Laurent expansion of $F(p)$ in the neighborhood of that point has the form $F(p) = \sum_{k=1}^{\infty} \frac{c_k}{p^k}$, the object function for $F(p)$ is

$$f(t) = \sum_{k=1}^{\infty} \frac{c_k}{(k-1)!} t^{k-1},$$

with the latter series converging for all t.

Example 16. Let us take the function $F(p) = p/(p^2 + 1)$. It is analytic in the neighborhood of the point at infinity and its Laurent series in the neighborhood of this point has the form

$$F(p) = \frac{1}{p^2\left(1+\frac{1}{p^2}\right)} = \frac{1}{p\left(1+\frac{1}{p^2}\right)}$$

$$= \frac{1}{p}\left(1 - \frac{1}{p^2} + \frac{1}{p^4} - \ldots + (-1)^n \frac{1}{p^{2n}} + \ldots\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{p^{2n+1}}.$$

Then

$$f(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} = \cos t,$$

which coincides with the well-known result.

Example 17. Find the result function of $f(t) = J_0(t)$, where $J_0(t)$ is the Bessel function of the first kind of order zero.

Solution. We know that

$$J_0(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(k!)^2 2^{2k}}.$$

Let us consider the function $F(p) = (1 + p^2)^{-1/2}$. It is single-valued when $|p| > 1$, analytic in the neighborhood of the point at infinity, and vanishes at that point. Let us find its Laurent expansion about that point by employing the binomial expansion formula:

$$F(p) \cdot (1 - p^2)^{-1/2} = \frac{1}{p}\left(1 + \frac{1}{p^2}\right)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{(k!)^2 2^{2k} p^{2k+1}}.$$

According to the first expansion theorem, the object function for $F(p)$ is

$$f(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(k!)^2 2^{2k}} = J_0(t).$$

Thus,

$$J_0(t) \doteqdot \frac{1}{\sqrt{1+p^2}}.$$

Example 18. Deduce the recursion formula

$$J_{n+1}(t) = \frac{2n}{t} J_n(t) - J_{n-1}(t).$$

Solution. We know (see Problem 596) that

$$J_n(t) \doteqdot \frac{(\sqrt{p^2+1}-p)^n}{\sqrt{p^2+1}}.$$

Using the property of integration of result function, we find that

$$\frac{J_n(t)}{p} \doteqdot \int_p^{\infty} \frac{(\sqrt{p^2+1}-p)^n}{\sqrt{p^2+1}}\, dp.$$

We put $(p^2+1)^{1/2} - p = v$. Then

$$\int \frac{(\sqrt{p^2+1}-p)^n}{\sqrt{p^2+1}}\, dp = -\int v^{n-1}\, dv = -\frac{v^n}{n} + c,$$

so that

$$\frac{J_n(t)}{t} \doteqdot -\frac{1}{n}\left[(\sqrt{p^2+1}-p)^n\right]\Big|_{p=p}^{p=\infty} = \frac{1}{n}(\sqrt{p^2+1}\,)^n. \tag{14.15}$$

On the other hand, from

$$J_{n-1}(t) \doteqdot \frac{1}{\sqrt{p^2+1}}(\sqrt{p^2+1}-p)^{n-1},$$

$$J_{n+1}(t) \doteqdot \frac{1}{\sqrt{p^2+1}}(\sqrt{p^2+1}-p)^{n+1}$$

we find that

$$J_{n-1}(t) + J_{n+1}(t) \doteq 2(\sqrt{p^2+1} - p)^n. \quad (14.16)$$

Combining (14.15) and (14.16) yields

$$J_{n-1}(t) - \frac{2n}{t} J_n(t) + J_{n+1}(t) = 0.$$

594. Show that

$$J_n(t) = \frac{J_{n-1}(t) - J_{n+1}(t)}{2} \quad (n = 1, 2, \ldots).$$

595. Find the result function of $f(t) = J_1(t)$.

596. Show that $J_n(t) \doteq \dfrac{(\sqrt{p^2+1} - p)^n}{\sqrt{p^2+1}}$.

597. Show that

$$t^{\frac{n}{2}} J_n(2\sqrt{t}) \doteq \frac{1}{p^{n+1}} e^{-1/p} \quad (n = 0, 1, 2, \ldots).$$

598. The modified Bessel function of order n is defined thus: $I_n(t) = i^{-n} J_n(it)$. Show that

$$I_n(t) \doteq \frac{(p - \sqrt{p^2-1})^n}{\sqrt{p^2-1}}.$$

599. Laguerre polynomials are defined by the generalized Rodrigues's formula thus:

$$L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n}(t^n e^{-t}) \quad (n = 0, 1, 2, \ldots).$$

Show that

$$L_n(t) \doteq \frac{1}{p}\left(1 - \frac{1}{p}\right)^n \quad (n = 0, 1, 2, \ldots).$$

600. Find the result function of $f(t) = \ln t$.

601. Show that $\operatorname{erf}(\sqrt{t}) \doteq \dfrac{1}{p\sqrt{p+1}}$, where

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du.$$

Suppose we need to know the sum of an infinite series of functions $\sum_{n=1}^{\infty} \varphi_n(t)$, where $\varphi_n(t)$ are object functions. By

going over to the respective result functions it is sometimes easier to find the new series than the initial one. If we then go back to an object function, we know the sum of the initial series.

Example 19. For Laguerre polynomials we have

$$L_n(t) \doteqdot \frac{1}{p}\left(1-\frac{1}{p}\right)^n.$$

Whence

$$\sum_{n=0}^{\infty}(-1)^n L_n(t) \doteqdot \frac{1}{p}\left[1-\left(1-\frac{1}{p}\right)+\left(1-\frac{1}{p}\right)^2-\ldots\right]$$

$$=\frac{1}{p}\cdot\frac{1}{1+1-\frac{1}{p}}=\frac{1}{2p-1}$$

$$=\frac{1}{2}\cdot\frac{1}{p-\frac{1}{2}} \doteqdot \frac{1}{2}e^{\frac{t}{2}}.$$

Thus,

$$\sum_{n=0}^{\infty}(-1)^n L_n(t)=\frac{1}{2}e^{\frac{t}{2}}.$$

602. Show that

$$\sum_{n=0}^{\infty}(-1)^n L_{2n}(2t)=\frac{1}{2}e^t(\sin t+\cos t).$$

603. Show that $\sum_{n=0}^{\infty}\frac{L_n(t)}{n!}=eJ_0(2\sqrt{t})$.

604. Show that $\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(t)=\frac{1}{2}\sin t$.

605. Show that $J_0(t)+2\sum_{n=1}^{\infty}J_{2n}(t)=1$, where $J_k(t)$ is the Bessel function of the first kind of order k.

606. Show that $\sum_{n=0}^{\infty}\frac{L_n(t)}{2^{n+1}}=e^{-t}$, $0<t<+\infty$.

Example 20. Evaluate the integral

$$f(t)=\int_0^{+\infty}\frac{\cos tu}{a^2+u^2}\,du,\quad t>0.$$

Solution. For $\cos tu$ the change-of-scale property yields

$$\cos tu \doteqdot \frac{p}{p^2+u^2}.$$

Whence

$$f(t)\doteqdot\int_0^{+\infty}\frac{p\,du}{(p^2+u^2)(a^2+u^2)}.$$

The integrand, if considered as a function of u, can be written thus:

$$\frac{p}{(p^2+u^2)(a^2+u^2)}=-\frac{p}{p^2-a^2}\,\frac{1}{p^2+u^2}+\frac{p}{p^2-a^2}\,\frac{1}{a^2+u^2}.$$

This means that

$$f(t)\doteqdot\left(-\frac{1}{p^2-a^2}\arctan\frac{u}{p}+\frac{p}{p^2-a^2}\,\frac{1}{a}\arctan\frac{u}{a}\right)\Bigg|_{u=0}^{u=+\infty}$$

$$=\frac{\pi}{2a}\,\frac{1}{p+a}.$$

If we go over to the object function, we have

$$f(t)=\frac{\pi}{2a}\,e^{-at}.$$

Evaluate the following integrals:

607. (a) $f(t)=\int_0^{\infty}\frac{u\sin tu}{1+u^2}\,du,\quad t>0;$

(b) $f(t)=\int_0^{\infty}\frac{\sin tu\cos u}{u}\,du,\quad t>0.$

608. Find the values of $f(t)$ and its first and second derivatives as $t\to+0$ if $f(t)\doteqdot\frac{p+1}{p(p^2+p+1)}$ and $f'(t)$, $f''(t)$, and $f'''(t)$ are object functions.

Finding the object function from the result function

The following two methods are used to find the object function $f(t)$ from the known result function $F(p)$:

I. If $F(p) = Q(p)/R(p)$ is a proper rational fraction, then the fraction is expressed as a sum of partial fractions and then the object function of each partial fraction is found using properties I-IX of the Laplace transform.

Example 21. Find the object function for $F(p) = \frac{1}{p(p-1)(p^2+4)}$.

Solution. We expand $F(p)$ in a sum of partial fractions:

$$\frac{1}{p(p-1)(p^2+4)} = \frac{A}{p} + \frac{B}{p-1} + \frac{Cp+D}{p^2+4}.$$

Finding the coefficients A, B, C, and D yields

$$F(p) \quad -\frac{1}{4}\frac{1}{p} + \frac{1}{5}\frac{1}{p-1} + \frac{1}{20}\frac{p}{p^2+4} - \frac{1}{5}\frac{1}{p^2+4}. \quad (14.17)$$

It now easy to find the object function for the partial fractions on the right-hand side of (14.17). Using the linearity property, we find that

$$f(t) \quad -\frac{1}{4} + \frac{1}{5}e^t + \frac{1}{20}\cos 2t - \frac{1}{10}\sin 2t.$$

Example 22. $F(p) \quad \frac{1}{(p^2+1)^2}$. Find the object function $f(t)$.

Solution. In the case at hand $F(p)$ is a partial fraction. To find the object function we use the property of convolution of object functions and the fact that

$$\frac{1}{p^2+1} \doteqdot \sin t.$$

We have

$$F(p) = \frac{1}{(p^2+1)} = \frac{1}{p^2+1}\,\frac{1}{p^2+1}$$

$$\doteqdot \int_0^t \sin(t-\tau)\sin\tau\,d\tau = \frac{1}{2}\int_0^t [\cos t - \cos(2\tau - t)]\,d\tau$$

$$= \frac{1}{2}t\cos t - \frac{1}{4}\sin(2\tau - t)\Big|_{\tau=0}^{\tau=t} = \frac{1}{2}t\cos t - \frac{1}{2}\sin t.$$

Example 23. $F(p) = \dfrac{e^{-p}}{p+1}$. Find the object function $f(t)$.

Solution. The fact that the result function has an exponential points to the necessity of using the property of translation of object function. Here $\tau = 1$ and $(p+1)^{-1} \doteqdot e^{-t}$, whence

$$\frac{e^{-p}}{p+1} \doteqdot e^{-(t-1)}\eta(t-1).$$

II. By using the second expansion theorem, which states that under certain conditions imposed on $F(p)$ the object function for $F(p)$ is

$$f(t) = \sum_{(p_k)} \operatorname{Res}[F(p)\,e^{pt}].$$

where the sum of residues is taken over all the singular points of $F(p)$.

For one, if $F(p) = Q(p)/R(p)$ is a proper rational fraction, the object function for $F(p)$ is

$$f(t) = \sum_{k=1}^{l} \frac{1}{(n_k-1)!} \lim_{p \to p_k} \frac{d^{n_k-1}}{dp^{n_k-1}} \{F(p)\,e^{pt}(p-p_k)^{n_k}\}, \tag{14.18}$$

where p_k are the poles of $F(p)$ of order n_k, and the summation in (14.18) is over all the poles of $F(p)$. If all the poles of $F(p)$ are simple, (14.18) simplifies:

$$f(t) = \sum_{k=1}^{l} \frac{Q(p_k)}{R'(p_k)} e^{p_k t}.$$

Example 24. $F(p) = \dfrac{p}{(p^2-1)^2}$. Find the object function $f(t)$.

Solution. The function $F(p)$ has poles at $p_1 = 1$ and $p_2 = -1$, each of order two. By formula (14.18) we have

$$f(t) = \lim_{p \to 1} \left[\frac{pe^{pt}}{(p+1)^2}\right]'_p + \lim_{p \to -1} \left[\frac{pe^{pt}}{(p-1)^2}\right]'_p = \frac{1}{2} t \sinh t.$$

Find the object functions for the given result functions and construct their graphs:

609. $F(p)=\frac{2p-p}{p^3}$. 610. $F(p)=\frac{e^{-2p}}{p^2}$.

611. $F(p)=\frac{e^{-2p}}{p-1}$. 612. $F(p)=\frac{e^{-3p}}{p+3}$.

Find the object functions for the following result functions:

613. $F(p)=\frac{1}{p^2+4p+5}$. 614. $F(p)=\frac{1}{p^2+4p+3}$.

615. $F(p)=\frac{p}{(p+1)^2}$. 616. $F(p)=\frac{p}{(p^2-1)^2}$.

617. $F(p)=\frac{1}{p+2p^2+p^3}$. 618. $F(p)=\frac{1}{7-p+p^2}$.

619. $F(p)=\frac{2p^3+p^2+2p+2}{p^5+2p^4+2p^3}$.

620. $F(p)=\frac{1}{p^2(p^2+1)}$.

621. $F(p)=\frac{p+2}{(p+1)(p-2)(p^2+4)}$.

622. $F(p)=\frac{n!}{p(p+1)(p+2)\dots(p+n)}$.

623. $F(p)=\frac{1}{p^4+2p^3+3p^2+2p+1}$.

624. $F(p)=\frac{p^2+2p-1}{p^3+3p^2+3p+1}$.

625. $F(p)=\frac{p}{p^2+1}$. 626. $F(p)=\frac{2p+3}{p^3+4p^2+5p}$.

627. $F(p)=\frac{1}{(p-1)^2(p+2)}$.

628. $F(p)=\frac{p^2+2p-1}{p^3-2p^2+2p-1}$.

629. $F(p)=\frac{3p^2}{(p^3-1)^2}$. 630. $F(p)=\frac{e^{-p}}{p^2-2p+5}+\frac{pe^{-2p}}{p^2+9}$.

631. $F(p)=\frac{e^{-3p}}{(p+1)^2}$. 632. $F(p)=\frac{e^{-p}}{p(p-1)}$.

633. $F(p)=\frac{1}{p^2+1}(e^{-2p}+2e^{-3p}+3e^{-4p})$.

634. $F(p)=\frac{e^{-p}}{p^2-1}+\frac{pe^{-2p}}{p^2-4}$.

635. $F(p)=\frac{e^{-p/2}}{p(p+1)(p^2+4)}$.

636. $F(p)=\frac{e^{-p}}{p^2}+\frac{2e^{-2p}}{p^3}+\frac{6e^{-3p}}{p^4}$.

637. $F(p)=\frac{e^{-p/2}}{p(p^2+1)}$.

Éfros's theorem

Let $f(t) \doteqdot F(p)$ and let $\Phi(p)$ and $q(p)$ be analytic and such that

$$\Phi(p)\, e^{-\tau q(p)} \doteqdot \varphi(t, \tau).$$

Then

$$F[q(p)]\,\Phi(p) \doteqdot \int_0^\infty f(\tau)\,\varphi(t, \tau)\,d\tau.$$

For one, if $\Phi(p) = p^{-1/2}$ and $q(p) = p^{1/2}$, then

$$\varphi(t, \tau)=\frac{1}{\sqrt{\pi t}}\,e^{-\tau^2/4t}.$$

For this reason, if we know that $F(p) \doteqdot f(t)$, Éfros's theorem allows to find the object function for $F(\sqrt{p})/\sqrt{p}$, namely,

$$\frac{F(\sqrt{p})}{\sqrt{p}} \doteqdot \frac{1}{\sqrt{\pi t}} \int_0^\infty f(\tau)\, e^{-\tau^2/4t}\, d\tau. \qquad (14.19)$$

Using Éfros's theorem, find the object functions for the following functions (a real):

638. $F(p)=\frac{e^{-\sqrt{p}x/a}}{p}$. 639. $F(p)=\frac{e^{-a\sqrt{p}}}{p\sqrt{p}}$.

640. $F(p)=\frac{e^{-\alpha\sqrt{p}}}{p^2}$. 641. $F(p)=\frac{e^{-\sqrt{p}x/a}}{\sqrt{p}\left(\frac{\sqrt{p}}{a}+h\right)}$.

642. $F(p)=\frac{e^{-\alpha\sqrt{p}}}{p(\sqrt{p}+a)}$.

Using Éfros's theorem, evaluate the following integrals:

643. $I(t)=\frac{1}{\sqrt{\pi t}}\int_0^\infty \cosh \tau e^{-\tau^2/4t}\,d\tau.$

644. $I(t)=\frac{1}{\sqrt{\pi t}}\int_0^\infty \cos \tau e^{-\tau^2/4t}\,d\tau.$

645. $I(t)=\frac{1}{\sqrt{\pi t}}\int_0^\infty \tau \sinh \tau e^{-\tau^2/4t}\,d\tau.$

646. $I(t)=\frac{1}{\sqrt{\pi t}}\int_0^\infty \tau \sin \tau e^{-\tau^2/4t}\,d\tau.$

§ 15 Solving the Cauchy Problem for Linear Ordinary Differential Equations with Constant Coefficients

Suppose we have a differential equation which for the sake of simplicity we assume to be of second order:

$$a_0\frac{d^2x}{dt^2}+a_1\frac{dx}{dt}+a_2x(t)=f(t), \tag{15.1}$$

where a_0, a_1, and a_2 are constants, $a_0\neq 0$, and $f(t)$ is an object function. We will look for the solution of Eq. (15.1) that satisfies the initial conditions (or data)

$$x(0)=x_0,\quad x'(0)=x_1. \tag{15.2}$$

Suppose $x(t)\doteq X(p)$ and $f(t)\doteq F(p)$. Applying the Laplace transformation to both sides of Eq. (15.1) and employing the properties of linearity and of differentiation of object function, instead of the differential equation (15.1) and the initial conditions (15.2) we have the *operator equation*

$$(a_0p^2+a_1p+a_2)X(p)-(a_0px_0+a_0x_1+a_1x_0)=F(p). \tag{15.3}$$

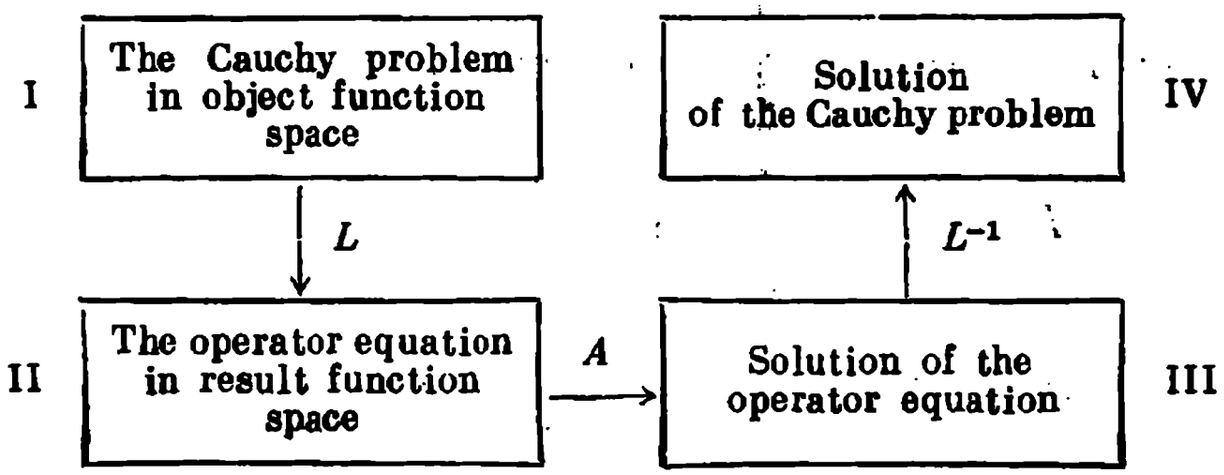
Equation (15.3) yields

$$X(p)=\frac{F(p)+a_0px_0+a_0x_1+a_1x_0}{a_0p^2+a_1p+a_2}.$$

This is the so-called *operator solution*. We can then find the object function $x(t)$, which is the solution of the Cauchy problem (15.1), (15.2).

Solving the Cauchy problem for an nth order differential equation does not, in principle, differ from the case with $n = 2$.

Solution of the Cauchy problem via the Laplace transformation



Here L stands for the Laplace transformation applied to I, A for the solution of the operator equation II, and L^{-1} for the inverse Laplace transformation applied to III.

Example 1. Solve the Cauchy problem

$$x'' + x = 2\cos t, \quad x(0) = 0, \quad x'(0) = -1.$$

Solution.

$$x(t) \doteqdot X(p), \quad x'(t) \doteqdot pX(p) - x(0) = pX(p),$$

$$x''(t) \doteqdot p^2X(p) - px(0) - x'(0) = p^2X(p) + 1, \quad \cos t \doteqdot \frac{p}{p^2+1},$$

so that the operator equation has the form

$$p^2X(p) + 1 + X(p) = \frac{2p}{p^2+1};$$

whence

$$X(p) = \frac{2p}{(p^2+1)^2} - \frac{1}{p^2+1}.$$

Now we have to find the object function of $X(p)$. The object function of $(p^2 + 1)^{-1}$ is $\sin t$. To find the object function of $2p/(p^2 + 1)^2$ we may use, for instance, the property of differentiation of result function (see § 14):

$$\frac{2p}{(p^2+1)^2} = -\left(\frac{1}{p^2+1}\right)'_p \doteqdot t\sin t.$$

This means that $X(p) \doteqdot t \sin t - \sin t = (t-1)\sin t$. Thus, $x(t) = (t-1)\sin t$.

Solve the following differential equations for the given initial conditions:

647. $x' + x = e^{-t}$, $x(0) = 1$.
648. $x' - x = 1$, $x(0) = -1$.
649. $x' + 2x = \sin t$, $x(0) = 0$.
650. $x'' = 1$, $x(0) = 0$, $x'(0) = 1$.
651. $x'' + x' = 1$, $x(0) = 0$, $x'(0) = 1$.
652. $x'' + x = 0$, $x(0) = 1$, $x'(0) = 0$.
653. $x'' + 3x' = e^t$, $x(0) = 0$, $x'(0) = -1$.
654. $x'' - 2x' = e^{2t}$, $x(0) = x'(0) = 0$.
655. $x'' + 2x' - 3x = e^{-t}$, $x(0) = 0$, $x'(0) = 1$.
656. $x''' + x' = 1$, $x(0) = x'(0) = x''(0) = 0$.
657. $x'' + 2x' = t \sin t$, $x(0) = 0$, $x'(0) = 0$.
658. $x'' + 2x' + x = \sin t$, $x(0) = 0$, $x'(0) = -1$.
659. $x''' - x'' = \sin t$, $x(0) = x'(0) = x''(0) = 0$.
660. $x''' + x' = t$, $x(0) = 0$, $x'(0) = -1$; $x''(0) = 0$.
661. $x'' - 2x' + x = e^t$; $x(0) = 0$, $x'(0) = 1$.
662. $x''' + 2x'' + 5x' = 0$, $x(0) = -1$, $x'(0) = 2$, $x''(0) = 0$.
663. $x'' - 2x' + 2x = 1$, $x(0) = x'(0) = 0$.
664. $x'' + x' = \cos t$, $x(0) = 2$, $x'(0) = 0$.
665. $x'' + 2x' + x = t^2$, $x(0) = 1$, $x'(0) = 0$.
666. $x''' + x'' = \sin t$, $x(0) = x'(0) = 1$, $x''(0) = 0$.
667. $x'' + x = \cos t$, $x(0) = -1$, $x'(0) = 1$,
668. $x''' + x'' = t$, $x(0) = -3$, $x'(0) = 1$, $x''(0) = 0$.
669. $x'' + 2x' + 5x = 3$, $x(0) = 1$, $x'(0) = 0$.
670. $x^{(IV)} - x'' = \cos t$, $x(0) = 0$, $x'(0) = -1$, $x''(0) = x'''(0) = 0$.
671. $x'' + 2x' + 2x = 1$, $x(0) = x'(0) = 0$.
672. $x'' + x = 1$, $x(0) = -1$, $x'(0) = 0$.
673. $x'' + 4x = t$, $x(0) = 1$, $x'(0) = 0$.
674. $x'' - 2x' + 5x = 1 - t$, $x(0) = x'(0) = 0$.
675. $x''' + x = 0$, $x(0) = 0$, $x'(0) = -1$, $x''(0) = 2$.
676. $x''' + x'' = \cos t$, $x(0) = -2$, $x'(0) = x''(0) = 0$.

677. $x''' + x' = e^t$, $x(0) = 0$, $x'(0) = 2$, $x''(0) = 0$.

678. $x^{(IV)} - x'' = 1$, $x(0) = x'(0) = x''(0) = x'''(0) = 0$.

679. $x'' + x' = \cos t$, $x(0) = 2$, $x'(0) = 0$.

680. $x'' - x' = te^t$, $x(0) = x'(0) = 0$.

681. $x''' + x' = \cos t$, $x(0) = 0$, $x'(0) = -2$, $x''(0) = 0$.

682. $x'' + 2x' + x = t$, $x(0) = x'(0) = 0$.

683. $x'' - x' + x = e^{-t}$, $x(0) = 0$, $x'(0) = 1$.

684. $x'' - x = \sin t$, $x(0) = -1$, $x'(0) = 0$.

685. $x''' + x = e^t$, $x(0) = 0$, $x'(0) = 2$, $x''(0) = 0$.

686. $x'' + x = 2 \sin t$, $x(0) = 1$, $x'(0) = -1$.

687. $x'' - 2x' + x = t - \sin t$, $x(0) = x'(0) = 0$.

688. $x'' + 2x' + x = 2 \cos^2 t$, $x(0) = x'(0) = 0$.

689. $x'' + 4x = 2 \cos t \cos 3t$, $x(0) = x'(0) = 0$.

690. $x'' + x = te^t + 4 \sin t$, $x(0) = x'(0) = 0$.

691. $x'' - x' = te^t$, $x(0) = 1$, $x'(0) = 0$.

692. $x'' + x' = 4 \sin^2 t$, $x(0) = 0$, $x'(0) = -1$.

693. $x''' - 2x'' + x' = 4$, $x(0) = 1$, $x'(0) = 2$, $x''(0) = -2$.

694. $x'' - 3x' + 2x = e^t$, $x(0) = x'(0) = 0$.

695. $x'' - x' = t^2$, $x(0) = 0$, $x'(0) = 1$.

696. $x''' + x = \frac{1}{2} t^2 e^t$, $x(0) = x'(0) = x''(0) = 0$.

697. $x'' + x = t \cos 2t$, $x(0) = x'(0) = 0$.

698. $x'' + n^2 x = a \sin(nt + \alpha)$, $x(0) = x'(0) = 0$.

699. $x''' + 6x'' + 11x' + 6x = 1 + t + t^2$, $x(0) = x'(0) = x''(0) = 0$.

700. $x^{(IV)} + 2x'' + x = t \sin t$, $x(0) = x'(0) = x''(0) = x'''(0) = 0$.

701. $x'' - 2\alpha x' + (\alpha^2 + \beta^2) x = 0$, $x(0) = 0$, $x'(0) = 1$.

702. $x'' + 4x = \sin t$, $x(0) = x'(0) = 0$.

703. $x''' + x' = e^{2t}$, $x(0) = x'(0) = x''(0) = 0$.

704. $x^{(IV)} + x''' = \cos t$, $x(0) = x'(0) = x''(0) = 0$, $x'''(0) = \gamma$.

705. $x'' - 4x = \sin \frac{3}{2} t \sin \frac{1}{2} t$, $x(0) = 1$, $x'(0) = 0$.

706. $x^{(IV)} - 5x'' + 10x' - 6x = 0$, $x(0) = 1$, $x'(0) = 0$, $x''(0) = 6$, $x''(0) = -14$.

707. $x'' + x' + x = te^t$, $x(0) = x'(0) = 0$.

708. $x'' + x = t \cos t$, $x(0) = x'(0) = 0$.

709. $x''' + 3x'' - 4x = 0$, $x(0) = x'(0) = 0$, $x''(0) = 2$.

710. $x''' + 3x'' + 3x' + x = 1$, $x(0) = x'(0) = x''(0) = 0$.

711. $x''' + x = 1$, $x(0) = x'(0) = x''(0) = 0$.

712. $x'' + \omega^2 x = a[\eta(t) - \eta(t - b)]$, $x(0) = x'(0) = 0$.

The requirement that the initial conditions must be fixed at $t = 0$ is unimportant, since by a linear change of variable t we can reduce a Cauchy problem with $t = t_0 \neq 0$ to that with the initial conditions at $t = 0$. Let us illustrate this with an example of a second-order differential equation.

We wish to find the solution of

$$a_0 x''(t) + a_1 x'(t) + a_2 x(t) = f(t) \qquad (15.4)$$

that satisfies the initial conditions $x(t_0) = x_0$ and $x'(t_0) = x_1$, with $t_0 \neq 0$. We put

$$t = \tau + t_0, \quad x(t) = x(\tau + t_0) = \tilde{x}(\tau), \quad f(t) = f(\tau + t_0) = \tilde{f}(\tau).$$

Then

$$x'(t) = x'(\tau + t_0) = \tilde{x}'(\tau),$$

$$x''(t) = x''(\tau + t_0) = \tilde{x}''(\tau),$$

and Eq. (15.4) and the initial conditions take the form

$$a_0 \tilde{x}''(\tau) + a_1 \tilde{x}'(\tau) + a_2 \tilde{x}(\tau) = \tilde{f}(\tau), \qquad (15.5)$$

$$\tilde{x}(0) = x_0, \quad \tilde{x}'(0) = x_1.$$

We have arrived at the Cauchy problem for Eq. (15.5) with initial conditions fixed at point $\tau = 0$.

Example 2. Find the solution of the equation $x''(t) + x'(t) = t$ that satisfies the initial conditions $x(1) = 1$, $x'(1) = 0$.

Solution. We put $t = \tau + 1$ and $x(t) = x(\tau + 1) = \tilde{x}(\tau)$. Then the equation and the initial conditions take the form

$$\tilde{x}''(\tau) + \tilde{x}'(\tau) = \tau + 1, \quad \tilde{x}(0) = 1, \quad \tilde{x}'(0) = 0, \qquad (15.6)$$

since $t = 1$ corresponds to $\tau = 0$. We can now construct

the operator equation for Eq. (15.6). Let $\tilde{x}(t) \doteq X(p)$. Then

$$\tilde{x}'(\tau) \doteq pX(p) - 1,$$
$$\tilde{x}''(\tau) \doteq p^2X(p) - p,$$

and the operator equation is

$$p^2X(p) - p + pX(p) - 1 = \frac{1}{p^2} + \frac{1}{p}.$$

Solving for $X(p)$ yields

$$X(p) = \frac{1}{p^3} + \frac{1}{p}.$$

If we go over to the object function, we find that

$$\tilde{x}(\tau) = 1 + \frac{\tau^2}{2}.$$

If we now substitute $t - 1$ for τ, we arrive at the sought solution of the Cauchy problem:

$$x(t) = 1 + \frac{(t-1)^2}{2}.$$

Solve the following differential equations with the given initial conditions:

713. $x'' + x = 0$, $x(\pi) = 1$, $x'(\pi) = 0$.

714. $x''(t) + x'(t) = 2t$; $x(1) = 1$, $x'(1) = -1$.

715. $x''(t) - x'(t) = -2t$, $x(2) = 8$, $x'(2) = 6$.

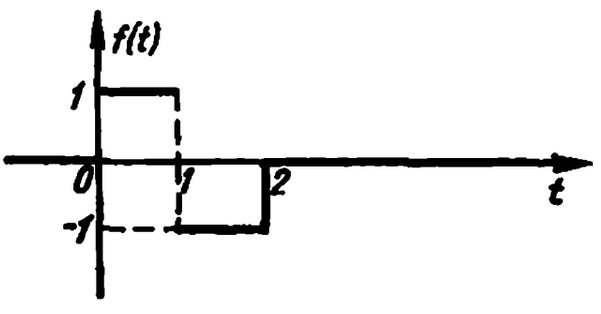
716. $x''(t) + x(t) = -2\sin t$; $x\left(\frac{\pi}{2}\right) = 0$, $x'\left(\frac{\pi}{2}\right) = 1$.

717. $x''(t) + 2x'(t) + x(t) = 2e^{1-t}$, $x(1) = 1$, $x'(1) = -1$.

Example 3. Solve the Cauchy problem

$$x'' + x = f(t), \quad x(0) = x'(0) = 0$$

if the function $f(t)$ is given in the form of the graph



Solution. Obviously,

$$f(t) = \eta(t) - 2\eta(t-1) + \eta(t-2),$$

whence, applying the formula

$$f(t-\tau) \doteqdot e^{-p\tau}F(p),$$

we obtain

$$f(t) \doteqdot \frac{1}{p} - 2\frac{e^{-p}}{p} + 1 \cdot e^{-2p}\frac{1}{p} = \frac{1-2e^{-p}+e^{-2p}}{p}.$$

Assuming that $x(t) \doteqdot X(p)$ and bearing in mind the initial conditions, we find that

$$x''(t) \doteqdot p^2X(p) - px(0) - x'(0) = p^2X(p).$$

The operator equation is

$$(p^2+1)X(p) = \frac{1-2e^{-p}+e^{-2p}}{p},$$

which means that

$$X(p) = \frac{1}{p(p^2+1)} - \frac{2e^{-p}}{p(p^2+1)} + \frac{e^{-2p}}{p(p^2+1)}.$$

Since

$$\frac{1}{p(p^2+1)} \doteqdot (1-\cos t)\,\eta(t),$$

due to the fact that

$$\frac{1}{p(p^2+1)} = \frac{1}{p} - \frac{p}{p^2+1},$$

we find that

$$\frac{1}{p(p^2+1)}e^{-p} \doteqdot [1-\cos(t-1)]\,\eta(t-1)$$

and

$$\frac{1}{p(p^2+1)}e^{-2p} \doteqdot [1-\cos(t-2)]\,\eta(t-2)$$

(we have used the property of translation of object function: if $f(t) \doteqdot F(p)$, then

$$f(t-\tau) \doteqdot e^{-p\tau}F(p)).$$

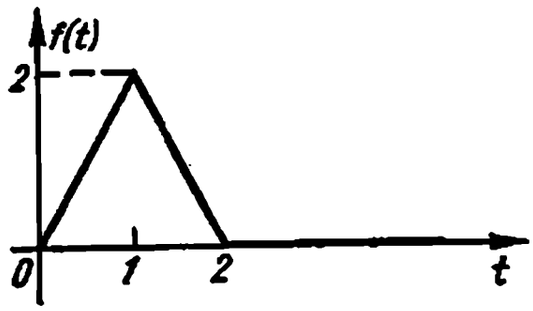
We have thus found that

$$x(t) = (1-\cos t)\,\eta(t) - 2[1-\cos(t-1)]\,\eta(t-1) + [1-\cos(t-2)]\,\eta(t-2)$$

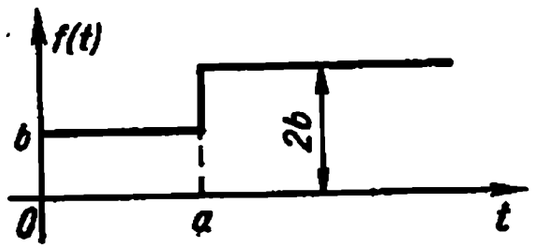
or

$$x(t) = 2\left[\sin^2\frac{t}{2}\,\eta(t) - 2\sin^2\frac{t-1}{2}\,\eta(t-1) + \sin^2\frac{t-2}{2}\,\eta(t-2)\right].$$

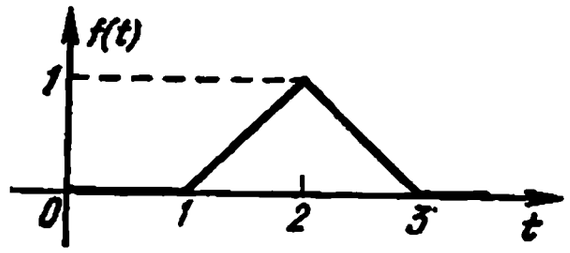
Solve the following Cauchy problems:

718. 

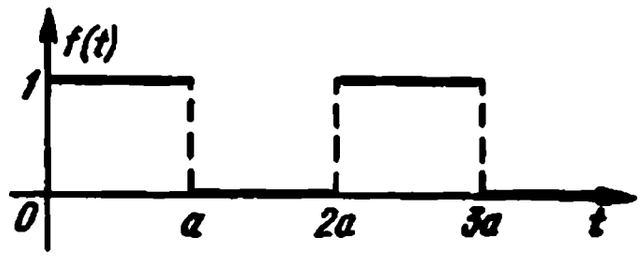
$$x'' + 4x = f(t),$$
$$x(0) = x'(0) = 0.$$

719. 

$$x'' + x = f(t),$$
$$x(0) = 1,\ x'(0) = 0.$$

720. 

$$x'' + 9x = f(t),$$
$$x(0) = 0,$$
$$x'(0) = 1.$$

721. 

$$x'' - 2x' + x = f(t),$$
$$x(0) = x'(0) = 0.$$

If we wish to use the operational method to solve physical problems involving time-dependent processes, we must do as follows:

(1) construct a differential or integro-differential equation (or a system of such equations) that describes the physical process being studied;

(2) the sought and given functions are assumed to be object functions and the corresponding result functions are introduced;

(3) construct the operator equation (or a system of such equations) corresponding to the initial equation (or the system of the initial equations);

(4) the operator equation is solved for the result function of the sought function;

(5) by employing the known properties of the Laplace transformation go over from the result function (or functions) to the object function (or functions), which is the answer to the problem;

(6) finally, analyze the answer from the standpoint of the physics of the problem.

Example 4. A particle of mass m is suspended by an elastic string of length l and elastic modulus λ and is at rest, with the string hanging vertical and not stretched. At $t = 0$ the suspension point O begins to oscillate vertically according to the law $\xi = a \sin \omega t$ with respect to the initial position and with the particle being not fixed. Find the law of motion of the particle.

Solution. Let us construct the differential equation that describes the motion of the particle. We send the x axis along the string downward, with the suspension point O taken as the origin of coordinates. Two forces act on the particle deflected from the equilibrium position: gravity mg and the tension of the string F_1. Hooke's law states that the latter is proportional to the string's elongation:

$$F_1 = \lambda \frac{x-(l+\xi)}{l},$$

where ξ and x are the displacements of suspension point O and particle from the initial position of point O. According to Newton's second law, the differential equation that describes the motion of the particle is

$$m \frac{d^2x}{dt^2} = mg - \lambda \frac{x-l-\xi}{l},$$

or

$$\frac{d^2x}{dt^2} + \frac{\lambda}{ml} x = g + \frac{\lambda}{m} + \frac{\lambda}{ml} \xi.$$

Introducing the notation $\mu^2 = \lambda/ml$ and bearing in mind that $\xi = a \sin \omega t$, we finally obtain

$$x''(t) + \mu^2 x(t) = g + \mu^2 l + a\mu^2 \sin \omega t. \qquad (15.7)$$

We will assume that the sought function $x(t)$ is an object function and that $X(p)$ is its result function, i.e. $x(t) \doteq X(p)$. Then

$$x''(t) \doteq p^2 X(p) - px(0) - x'(0),$$

and since $x(0) = l$ and $x'(0) = 0$, we find that

$$x''(t) \doteq p^2X(p) - lp.$$

The result functions of the functions on the right-hand side of Eq. (15.7) are known:

$$g + \mu^2 l \doteq \frac{g+\mu^2 l}{p}, \quad \sin \omega t \doteq \frac{\omega}{p^2+\omega^2}.$$

Now we construct the operator equation corresponding to the differential equation (15.7). We have

$$p^2X(p) - lp + \mu^2 X(p) = \frac{g+\mu^2 l}{p} + \frac{a\mu^2\omega}{p^2+\omega^2}.$$

From this linear algebraic equation of the first degree we can easily find the result function $X(p)$ of the sought object function $x(t)$:

$$X(p) = l\frac{p}{p^2+\mu^2} + \frac{g+l\mu^2}{p(p^2+\mu^2)} + \frac{a\mu^2\omega}{(p^2+\mu^2)(p^2+\omega^2)},$$

or

$$X(p) = l\frac{p}{p^2+\mu^2} + \frac{g+l\mu^2}{\mu^2}\left(\frac{1}{p} - \frac{p}{p^2+\mu^2}\right) + \frac{a\mu^2\omega}{\omega^2-\mu^2}\left(\frac{1}{p^2+\mu^2} - \frac{1}{p^2+\omega^2}\right), \tag{15.8}$$

provided $\mu \neq \omega$.

Using the rules of operational calculus, we can find the object function that describes the motion of the particle for $\mu \neq \omega$:

$$\begin{aligned} x(t) &= l\cos\omega t + \frac{g+l\mu^2}{\mu^2}(1-\cos\omega t) \\ &\quad + \frac{a\mu^2\omega}{\omega^2-\mu^2}\left(\frac{\sin\mu t}{\mu} - \frac{\sin\omega t}{\omega}\right) \\ &= l\cos\omega t + \frac{g+l\mu^2}{\mu^2} - \frac{g}{\mu^2}\cos\omega t - l\cos\omega t \\ &\quad + \frac{a\mu}{\omega^2-\mu^2}(\omega\sin\mu t - \mu\sin\omega t) \\ &= \frac{a\mu}{\omega^2-\mu^2}(\omega\sin\mu t - \mu\sin\omega t) + \frac{g+l\mu^2}{\mu^2} - \frac{g}{\mu^2}\cos\omega t. \end{aligned}$$

To find the law of motion of the particle for $\mu = \omega$, we will find the limit of $x(t)$ as $\mu \to \omega$ by employing L'Hospi-

tal's rule. We have

$$\lim_{\mu\to\omega} x(t) = a\omega \lim_{\mu\to\omega} \frac{\omega \sin \mu t - \mu \sin \omega t}{\omega^2 - \mu^2} + \frac{g + \omega^2 l}{\omega^2} - \frac{g}{\omega^2} \cos \omega t$$

$$= a\omega \lim_{\mu\to\omega} \frac{\omega t \cos \mu t - \sin \omega t}{-2\mu} + \frac{g + \omega^2 l}{\omega^2} - \frac{g}{\omega^2} \cos \omega t$$

$$= \frac{a}{2} (\omega t \cos \omega t - \sin \omega t) + \frac{g + \omega^2 l}{\omega^2} - \frac{g}{\omega^2} \cos \omega t.$$

Thus, the law of motion of the particle is

$$x(t) = \begin{cases} \dfrac{a\mu}{\omega^2 - \mu^2} (\omega \sin \mu t - \mu \sin \omega t) \\ \quad + \dfrac{g + \mu^2 l}{\mu^2} - \dfrac{g}{\mu^2} \cos \omega t, \quad \mu \neq \omega; \\ \dfrac{a}{2} (\omega t \cos \omega t - \sin \omega t) + \dfrac{g + \omega^2 l}{\omega^2} - \dfrac{g}{\omega^2} \cos \omega t, \quad \mu = \omega. \end{cases}$$

Example 5. Find the electric current $i = i(t)$ in a circuit that consists of a coil of inductance L, a resistor R, and a capacitor C connected in series. A DC source is connected to the circuit, with the voltage being U_0. At the initial moment the electric current and the charge on the capacitor are zero.

Solution. We construct an equation that describes the processes in the circuit. We know that the potential drops on the inductance coil, U_L, on the resistor, U_R, and on the capacitor, U_C, are related to the electric current $i(t)$ thus:

$$U_L = L \frac{di(t)}{dt}, \quad U_R = Ri(t), \quad U_C = \frac{1}{C} \int_0^t i(\tau)\, d\tau.$$

By Kirchhoff's voltage law,

$$U_L + U_R + U_C - U_0 = 0.$$

Whence

$$L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t i(\tau)\, d\tau = U_0, \tag{15.9}$$

which is an integro-differential equation.

Let us assume that the electric current $i(t)$ is an object function whose result function is $J(p)$, i.e. $i(t) \doteq J(p)$. Then

$$\frac{di(t)}{dt} \doteq pJ(p) - i(0) = pJ(p),$$

since $i(0) = 0$, and

$$\int_0^t i(\tau)\, d\tau \doteq \frac{J(p)}{p}.$$

The operator equation that corresponds to the integro-differential equation (15.9) is

$$LpJ(p) + RJ(p) + \frac{1}{C}\frac{J(p)}{p} = \frac{U_0}{p}.$$

Solving for $J(p)$, we obtain

$$J(p) = \frac{CU_0}{LCp^2 + CRp + 1} = \frac{\frac{U_0}{L}}{p^2 + \frac{R}{L}p + \frac{1}{CL}}.$$

If we isolate a perfect square in the denominator, we can write

$$J(p) = \frac{\frac{U_0}{L}}{\left(p + \frac{R}{2L}\right)^2 + \frac{1}{CL} - \frac{R^2}{4L^2}}. \qquad (15.10)$$

The result function (15.10) gives us the object function, the electric current $i(t)$, if we employ the properties of the Laplace transformation. We have

$$i(t) = \frac{U_0}{L\sqrt{\frac{1}{CL} - \frac{R^2}{4L^2}}} e^{-\frac{R}{2L}t} \sin \sqrt{\frac{1}{CL} - \frac{R^2}{4L^2}}\, t,$$

$$4L > CR^2;$$

$$i(t) = \frac{U_0}{L} te^{-\frac{R}{2L}t}, \quad 4L = CR^2;$$

$$i(t) = \frac{U_0}{L\sqrt{\frac{R^2}{4L^2} - \frac{1}{CL}}} e^{-\frac{R}{2L}t} \sinh \sqrt{\frac{R^2}{4L^2} - \frac{1}{CL}}\, t,$$

$$4L < CR^2.$$

722. A particle of mass m is moving along a straight line under a restoring force proportional to displacement, $m\lambda x$, and a resisting force proportional to velocity, $2m\mu v$. At time

zero, $t = 0$, the particle was at a distance x_0 from the equilibrium position and its velocity was v_0. Show that if $n^2 = \lambda^2 - \mu^2$, the particle displacement is given by the formula

$$\frac{1}{n} e^{-\mu t} [nx_0 \cos nt + (v_0 + \mu x_0) \sin nt].$$

723. A particle of mass m can perform small oscillations about the equilibrium position and is under a restoring force proportional to displacement, mn^2x. A constant force F acting over a period T takes the particle out of equilibrium. Show that the oscillation amplitude is

$$\frac{2F}{mn^2} \sin \frac{nT}{2} \quad \text{for } t > T.$$

724. A simple pendulum of length l is taken out of equilibrium by small displacements of its suspension point in the horizontal direction. Show that if the suspension point is shifted by a distance a, the displacement of the pendulum will be $a(1 - \cos nt)$, with $n^2 = g/l$.

725. A particle of mass m is thrown upward with a velocity v_0. The two forces on the particle are gravity and resisting force $2kmv$. Show that at time $t = 0$ the particle will be at a distance $-\frac{gt}{2k} + \frac{g + 2kv_0}{4k^2} (1 - e^{-2kt})$ from the point where it was thrown.

726. A 2-gram particle is moving in a straight line under force F that increases by a dynes every second. At time zero, $t = 0$, the particle was at the origin of coordinates and had a velocity $v_0 = 10$ cm/s. Knowing that the initial magnitude of the force was $F_0 = 4$ dynes and that the particle velocity at a distance of 450 cm from the origin of coordinates was $v = 105$ cm/s, find the value of a.

727. A particle of mass m is moving in a straight line under a repulsive force F (source of force at origin of coordinates O) directly proportional to the distance, $F = 4mx$, and a resisting force $R = 3mv$. At time zero the particle was at a distance 1 from the origin of coordinates and its velocity was zero. Find the equation of motion of the particle.

728. A particle of mass m is falling in a medium that exerts a resisting force proportional to the first power of the

particle velocity. Find the maximum velocity of the particle if it is known that at $v = 1$ m/s the resisting force is one third of the particle's weight and the initial velocity v_0 is zero.

729. A particle of mass m is moving in a medium that exerts a resisting force proportional to the first power of the velocity (the proportionality factor is k). What distance will the particle travel before it stops if the velocity imparted to it is v_0 and there are no other forces except the resisting force acting on it?

730. A heavy homogeneous chain of mass m and length $2l$ is lying on a smooth horizontal table with half of it hanging off the side of the table. Determine the equation of motion of the chain as it slips off the table and find the duration of the slip.

731. A particle of mass m lies on the straight line connecting two point sources of force, A and B, with the distance between these forces being $2d$. Each source attracts the particle with a force proportional to the distance between source and particle (the proportionality factor mk^2 is the same for both sources). At time zero the particle lies at a distance a from the middle O of segment AB with no initial velocity. Find the equation of motion of the particle.

732. A fixed center O attracts a particle of mass m with a force $F = \mu mr$, where r is the distance between particle and center, and μ a constant factor. At time zero, $r = a$ and $v = 0$. When will the particle arrive at O?

733. Suppose the velocity imparted to a boat is $v_0 = = 6m/s$. After 69 s the velocity fell off by one-half. Find the equation of motion of the boat if the resisting force exerted by the water is directly proportional to the boat's velocity.

734. A particle of mass $m = 2$ is oscillating along the x axis under a restoring force that is proportional to the distance between the point and the origin of coordinates O (the proportionality factor is 8) and a perturbing force $F = = 4 \cos t$. Find the equation of motion of the particle if $x = 0$ and $v = 0$ at $t = 0$.

735. Determine the equation of motion of a particle of mass m that is attracted to a fixed center O with a force directly proportional to the distance between particle and center and equal to k^2m at a distance of unit magnitude. At

time zero the particle was at a distance a from O and had a velocity v_0 directed at right angles to the straight line connecting O and the initial position of the particle.

736. Solve Problem 735 assuming that the particle is repelled by the center O with a force directly proportional to its distance from O (the proportionality factor is the same).

737. An emf $E \cos(\omega t + \alpha)$ is applied at $t = 0$ to a circuit consisting of a capacitance C and an inductance L connected in series. The initial current and charge are zero. Show that at time $t \neq 0$ the current will be

$$E\{\omega \sin(\omega t + \alpha) - n \cos\alpha \sin nt - \omega \sin\alpha \cos nt\} \frac{1}{L(\omega^2 - n^2)},$$

where $n^2 = 1/LC$, and we assume that $n^2 \neq \omega^2$.

738. Take the circuit of Problem 737, with zero current and charge at $t = 0$, and apply an emf $E \sin nt$ in resonance with the circuit. Show that the current at $t \neq 0$ is $(E/2L)\, t \sin nt$, with $n^2 = 1/LC$.

739. An emf $E \sin(\omega t + \alpha)$ is applied to a circuit consisting of a resistor R with inductance L. The initial current is zero. Show that the current at $t \neq 0$ is

$$E\{\sin(\gamma - \alpha)\, e^{-\frac{Rt}{L}} + \sin(\omega t + \alpha - \gamma)\}(R^2 \quad L^2\omega^2)^{-\frac{1}{2}},$$

where $\tan\gamma = \omega L/R$.

740. En emf equal to E_1 when $0 < t \leqslant T$ and E_2 when $t > T$ (E_1 and E_2 are constant) is applied to a series circuit consisting of an inductance L, resistance R, and capacitance C. The initial charge and current are zero. Show that when $t > T$, the current in the circuit is

$$\frac{E_1}{nL} e^{-\mu t} \sin nt - \frac{E_1 - E_2}{nL} e^{-\mu(t-T)} \sin n(t-T),$$

where $\mu = R/2L$, and $n^2 = 1/LC - R^2/4L^2$; it is assumed that n^2 is positive.

741. A constant emf E is applied to a series circuit consisting of an inductance L, resistance R, and capacitance C. The initial charge and current are zero. Show that at $t \neq 0$

the current I is given by the following formula:

$$I = \begin{cases} \dfrac{E}{nL} e^{-\mu t} \sin nt & \text{at } n^2 > 0, \\ \dfrac{E}{L} te^{-\mu t} & \text{at } n^2 = 0, \end{cases}$$

where $n^2 = 1/LC - R^2/4L^2$, and $\mu = R/2L$.

Solving some linear ordinary differential equation with varying constants

Suppose we are dealing with the equation

$$a_0(t)\, x^{(n)}(t) + a_1(t)\, x^{(n-1)}(t) + \ldots + a_n(t)\, x(t) = f(t), \tag{15.11}$$

where $a_0(t)$, $a_1(t)$, . . ., $a_n(t)$ are polynomials in t of degrees not less than m, and $f(t)$ is an object function. Let us assume that the Cauchy problem for Eq. (15.11) with

$$x|_{t=0} = x_0,\ x'|_{t=0} = x_0',\ \ldots,\ x^{(n-1)}|_{t=0} = x_0^{(n-1)} \tag{15.12}$$

has a solution, and $x(t) \doteqdot X(p)$. By the property of differentiation of result function,

$$t^k x^{(s)}(t) \doteqdot (-1)^k \frac{d^k}{dp^k} \mathcal{L}\{x^{(s)}(t)\}$$

$$= (-1)^k \frac{d^k}{dp^k} [p^s X(p) - p^{s-1}x_0 - \ldots - x_0^{(s-1)}].$$

Here $\mathcal{L}\{x^{(s)}(t)\}$ is the result function (or Laplace transform) of $x^{(s)}(t)$. Thus, if we apply the Laplace transformation to both sides of Eq. (15.11), we have an mth order ordinary differential equation with respect to the result function $X(p)$ of $x(t)$. If $m < n$, the problem of integrating Eq. (15.11) is simplified.

Example 6. Find the general solution of

$$tx'' - 2x' = 0. \tag{15.13}$$

Solution. Let $x(t) \doteqdot X(p)$. Then

$$x'(t) \doteqdot pX(p) - x(0),$$

$$x''(t) \doteqdot p^2X(p) - px(0) - x'(0),$$

$$tx''(t) \doteq -\frac{d}{dp}[p^2X(p) - px(0) - x'(0)]$$

$$= -p^2\frac{dX(p)}{dp} - 2pX(p) + x(0).$$

Equation (15.13) then takes the form

$$-p^2\frac{dX(p)}{dp} - 2pX(p) + x(0) - 2pX(p) + 2x(0) = 0,$$

or

$$\frac{dX(p)}{dp} + \frac{4}{p}X(p) = \frac{3x(0)}{p^2}.$$

This is a linear nonhomogeneous equation with respect to $X(p)$. Integrating it, we find that

$$X(p) = \frac{x(0)}{p} + \frac{C_1}{p^4},$$

from which it follows that

$$x(t) = x(0) + C_1\frac{t^3}{3!}$$

is the solution of Eq. (15.13).

Example 7. Let us take *Bessel's differential equation*

$$t^2x''(t) + tx'(t) + (t^2 - n^2)x(t) = 0$$
$$(t > 0, \text{ and } n \text{ an integer}) \qquad (15.14)$$

and look for its solution that satisfies the initial conditions $x(0) = x_0$, $x'(0) = x_1$. Let $x(t) \doteq X(p)$. Then

$$x'(t) \doteq pX(p) - x_0, \quad x''(t) \doteq p^2X(p) - px_0 - x_1.$$

Moreover,

$$t^2x''(t) \doteq \frac{d^2}{dp^2}[p^2X(p) - px_0 - x_1] = \frac{d^2}{dp^2}[p^2X(p)],$$

$$tx'(t) \doteq -\frac{d}{dp}[pX(p) - x_0] = -\frac{d}{dp}[pX(p)],$$

and Eq. (15.14) is transformed thus:

$$\frac{d^2}{dp^2}[p^2X(p)] - \frac{d}{dp}[pX(p)] + \frac{d^2X(p)}{dp^2} - n^2X(p) = 0,$$

or

$$(1 + p^2)\frac{d^2X(p)}{dp^2} + 3p\frac{dX(p)}{dp} + (1 - n^2)X(p) = 0. \qquad (15.15)$$

To solve Eq. (15.15) we introduce a new variable and a new result function according to the following formulas:

$$p = \sinh u, \quad X(p) = z/\cosh u.$$

Instead of Eq. (15.15) we now have

$$\frac{d^2z}{du^2} - n^2z = 0.$$

Its general solution is

$$z = C_1e^{nu} + C_2e^{-nu}.$$

Since $p = \sinh u$, we obtain $\cosh u = \sqrt{p^2 + 1}$. Bearing in mind the expressions for $\sinh u$ and $\cosh u$ in terms of the exponential function, we find that

$$\frac{e^u - e^{-u}}{2} = p, \quad \frac{e^u + e^{-u}}{2} = \sqrt{p^2 + 1},$$

whence

$$e^u = \sqrt{p^2 + 1} + p, \quad e^{-u} = \sqrt{p^2 + 1} - p,$$

so that

$$z = C_1(\sqrt{p^2 + 1} + p)^n + C_2(\sqrt{p^2 + 1} - p)^n.$$

For $X(p)$ we then have

$$X(p) = \frac{z}{\cosh u} = \frac{C_1(\sqrt{p^2 + 1} + p)^n + C_2(\sqrt{p^2 + 1} - p)^n}{\sqrt{p^2 + 1}}. \quad (15.16)$$

For $n = 0$ this yields

$$X(p) = \frac{C_1 + C_2}{\sqrt{p^2 + 1}} \doteq (C_1 + C_2) J_0(t).$$

If we put $C_1 + C_2 = 1$, we obtain the solution $x(t) = J_0(t)$, the Bessel function of the first kind of order zero. Putting $n = 1$ and assuming that $C_1 = 0$ and $C_2 = 1$, we obtain the solution $x(t) = J_1(t)$, the Bessel function of the first kind of order one.

Find the solutions of the following equations:

742. $tx'' + (2t - 1)x' + (t - 1)x = 0$.

743. $tx'' + 2x' = 0$.

744. $x'' + (t + 1)x' + tx = 0$, $x(0) = 1$, $x'(0) = -1$.

745. $x'' + (t + b)x' = 0$, $x(0) = -1$, $x'(0) = 0$ (b is any real number).

746. $x'' + tx' - (t + 1)x = 0$, $x(0) = x'(0) = 1$.

747. $x'' - tx' + nx = 0$, n is real and positive (the

Chebyshev-Hermite equation), with
(a) $x(0) = 1, \quad x'(0) = 0, \quad n = 2k;$
(b) $x(0) = 0, \quad x'(0) = 1, \quad n = 2k + 1.$

§ 16 Duhamel's Integral

If $f(t)$ is continuous in $[0, +\infty)$ and $\varphi(t)$ is continuously differentiable in $[0, +\infty)$ and

$$F(p) \doteq f(t), \quad \Phi(p) \doteq \varphi(t),$$

then

$$F(p)\,\Phi(p) \doteq \int_0^t f(\tau)\,\varphi(t-\tau)\,d\tau.$$

By the property of differentiation of object function,

$$pF(p)\,\Phi(p) \doteq f(t)\,\varphi(0) + \int_0^t f(\tau)\,\varphi'(t-\tau)\,d\tau. \qquad (16.1)$$

This is known as *Duhamel's formula.*

Suppose we have to solve a linear ordinary differential equation of order n with constant coefficients

$$L[x] \equiv a_0 x^{(n)}(t) + a_1 x^{(n-1)}(t) + \ldots + a_n x(t) = f(t), \quad a_0 \neq 0. \qquad (16.2)$$

with zero initial conditions,

$$x(0) = x'(0) = \ldots = x^{(n-1)}(0) = 0. \qquad (16.3)$$

Let us assume that we know the solution of equation

$$L[x] = 1, \qquad (16.4)$$

which has the same left-hand side and a right-hand side equal to unity, with the same initial conditions (16.3). If we go over to operator equations, we obtain

$$A(p)\,X(p) = F(p) \qquad (16.5)$$

for (16.2) and

$$A(p)\,X_1(p) = \frac{1}{p} \qquad (16.6)$$

for (16.4) (here $A(p)$ is a known polynomial in p). From (16.5) it follows that $X(p) = F(p)/A(p)$ and from (16.6) that $A(p) = 1/pX_1(p)$, whence $X(p) = pX_1(p)\,F(p)$. Ac-

cording to (16.1) we have

$$pX_1(p)\,F(p) \doteq f(t)\,x_1(0) + \int_0^t f(\tau)\,x_1'(t-\tau)\,d\tau. \quad (16.7)$$

Bearing in mind that $x_1(0) = 0$, we find that

$$X(p) = pX_1(p)\,F(p) \doteq \int_0^t f(\tau)\,x_1'(t-\tau)\,d\tau. \quad (16.8)$$

The solution $x(t)$ of Eq. (16.2) with zero initial conditions (16.3) will then be

$$x(t) = \int_0^t f(\tau)\,x_1'(t-\tau)\,d\tau, \quad (16.9)$$

where $x_1(t)$ is the solution of the problem (16.4), (16.3).

Example 1. Use Duhamel's formula to solve the following equation with zero initial conditions:

$$x''(t) - x(t) \quad \frac{1}{1+e^t}, \quad x(0) = x'(0) = 0.$$

Solution. Consider the auxiliary problem

$$x_1''(t) - x_1(t) = 1, \quad x_1(0) = x_1'(0) = 0.$$

Applying the operational method, we find that

$$X_1(p) = \frac{1}{p(p^2-1)},$$

whence

$$x_1(t) = \int_0^t \sinh\tau\,d\tau = \cosh t - 1.$$

By (16.9),

$$x(t) = \int_0^t \frac{1}{1+e^\tau} \sinh(t-\tau)\,d\tau$$

$$= \frac{1}{2}(e^t - te^t - 1) + \sinh t \ln\frac{1-e^t}{2}.$$

The requirement that the initial conditions be zero is unimportant, since a simple change to a new unknown function reduces the problem with nonzero initial conditions

to that with zero initial conditions. Let us illustrate this by an example of a second-order differential equation.

Suppose we wish to find the solution of

$$a_0x''(t) + a_1x'(t) + a_2x(t) = f(t) \qquad (16.10)$$

that satisfies the nonzero initial conditions

$$x(0) = x_0, \quad x'(0) = x_1.$$

We put

$$y(t) = x(t) - x_0 - x_1t. \qquad (16.11)$$

Then

$$y'(t) = x'(t) - x_1, \quad y''(t) = x''(t),$$

and Eq. (16.10) becomes

$$a_0y''(t) + a_1y'(t) + a_2y(t) = f_1(t),$$

where

$$f_1(t) = f(t) - a_1x_1 - a_2x_0 - a_2x_1t.$$

Moreover, in view of (16.11) we have

$$y(0) = x(0) - x_0 = 0, \quad y'(0) = x'(0) - x_1 = 0.$$

Thus, we have arrived at the following Cauchy problem:

$$a_0y''(t) + a_1y'(t) + a_2y(t) = f_1(t),$$
$$y(0) = 0, \quad y'(0) = 0$$

with zero initial conditions.

Example 2. Use Duhamel's formula to solve the following Cauchy problem:

$$x'' + 2x' + x = \frac{e^{-t}}{(1+t)^2} + t, \qquad (16.12)$$

$$x(0) = -2, \quad x'(0) = 1. \qquad (16.13)$$

Solution. We reduce the problem (16.12), (16.13) to that with zero initial conditions. To this end we assume that

$$y(t) = x(t) + 2 - t.$$

Then

$$y'(t) = x'(t) - 1, \quad y''(t) = x''(t),$$

and Eq. (16.12) becomes

$$y''(t) + 2y'(t) + y(t) = \frac{e^{-t}}{(1+t)^2},$$

where

$$y(0) = 0, \quad y'(0) = 0.$$

Solving the latter via Duhamel's formula, we find that

$$y = e^{-t}[t - \ln(1 + t)].$$

The solution of the problem (16.12), (16.13) is

$$x(t) = e^{-t}[t - \ln(1 + t)] - 2 + t.$$

Using Duhamel's formula, find the solutions of the following equations that satisfy the given initial conditions:

748. $x'' - x' = \dfrac{e^{2t}}{(1+e^t)^2}$, $x(0) = x'(0) = 0$.

749. $x'' + 2x' + x = \dfrac{e^{-t}}{1+t}$, $x(0) = x'(0) = 0$.

750. $x'' - x' = \dfrac{e^{2t}}{2+e^t}$, $x(0) = x'(0) = 0$.

751. $x'' - x' = \dfrac{1}{1+e^t}$, $x(0) = x'(0) = 0$.

752. $x'' + x = \dfrac{1}{2+\cos t}$, $x(0) = x'(0) = 0$.

753. $x'' + x = \dfrac{1}{4+\tan^2 t}$, $x(0) = x'(0) = 0$.

754. $x'' + x = \dfrac{1}{1+\cos^2 t}$, $x(0) = x'(0) = 0$.

755. $x'' + x = \dfrac{1}{1+\sin^2 t}$, $x(0) = x'(0) = 0$.

756. $x'' - x = \tanh t$, $x(0) = x'(0) = 0$.

757. $x''' + x' = \dfrac{1}{2+\sin t}$, $x(0) = x'(0) = x''(0) = 0$.

§ 17 Solving Systems of Linear Differential Equations by the Operational Method

Solution of systems of linear ordinary differential equations with constant coefficients by the operational method follows the same lines as that of one differential equation. Suppose, for instance, we have to solve a system of second-order differ-

ential equations

$$\sum_{k=1}^{n}\left(a_{ik}\frac{d^2x_k}{dt^2}+b_{ik}\frac{dx_k}{dt}+c_{ik}x_k\right)=f_i(t) \qquad (17.1)$$

$$(i=1, 2, \ldots, n),$$

with a_{ik}, b_{ik} and c_{ik} constant, with the following initial conditions:

$$x_k(0)=\alpha_k, \quad x_k'(0)=\beta_k. \qquad (17.2)$$

If we denote the result functions of $x_k(t)$ and $f_i(t)$ by $X_k(p)$ and $F_i(p)$, respectively, we can go over from the system (17.1) together with (17.2) to the system of operator equations

$$\sum_{k=1}^{n}(a_{ik}p^2+b_{ik}p+c_{ik})X_k(p)$$

$$=F_i(p)+\sum_{k=1}^{n}[(a_{ik}p+b_{ik})\alpha_k+a_{ik}\beta_k] \qquad (17.3)$$

$$(i=1, 2, \ldots, n).$$

Solving system (17.3) as a system of linear algebraic equations in the unknowns $X_k(p)$, we will find these unknowns and then the object functions $x_k(t)$ $(k=1, 2, \ldots, n)$. The latter will be the solutions of the Cauchy problem (17.1), (17.2).

Example. Solve the system of equations

$$\begin{cases} x''=3(y-x+z), \\ y''=x-y, \\ z''=-z, \end{cases}$$

$$x(0)=x'(0)=0,$$

$$y(0)=0, \quad y'(0)=-1,$$

$$z(0)=1, \quad z'(0)=0.$$

Solution. Going over to the system of operator equations yields

$$\begin{cases} p^2X=3(Y-X+Z), \\ p^2Y+1=X-Y, \\ p^2Z-p=-Z, \end{cases}$$

where

$$X(p) \doteq x(t), \quad Y(p) \doteq y(t); \quad Z(p) \doteq z(t).$$

Solving the latter system for $X(p)$, $Y(p)$, and $Z(p)$, we obtain

$$X(p) = \frac{3(p-1)}{p^2(p^2+4)}, \quad Y(p) = \frac{3(p-1)}{p^2(p^2+1)(p^2+4)} - \frac{1}{p^2+1},$$

$$Z(p) = \frac{p}{p^2+1}.$$

We can now easily find the object functions for $X(p)$, $Y(p)$, and $Z(p)$. They are

$$x(t) = \frac{3}{4}(1-t) - \frac{3}{4}\cos 2t + \frac{3}{8}\sin 2t,$$

$$y(t) = \frac{3}{4}(1-t) + \frac{1}{4}\cos 2t - \frac{1}{8}\sin 2t - \cos t,$$

$$z(t) = \cos t.$$

Find the solutions of the following systems of equations:

758. $\begin{cases} x' + y = 0, \\ y' + x = 0, \end{cases} \quad x(0) = 1, \ y(0) = -1.$

759. $\begin{cases} x + x' = y + e^t, \\ y + y' = x + e^t, \end{cases} \quad x(0) = y(0) = 1.$

760. $\begin{cases} x' - y' - 2x + 2y = 1 - 2t, \\ x'' + 2y' + x = 0, \end{cases} \quad x(0) = y(0) = x'(0) = 0.$

761. $\begin{cases} x'' - 3x' + 2x + y' - y = 0, \\ -x' + x + y'' - 5y' + 4y = 0, \end{cases}$

$$x(0) = x'(0) = y'(0) = 0, \ y(0) = 1.$$

762. $\begin{cases} x' = -y, \\ y' = 2x + 2y, \end{cases} \quad x(0) = y(0) = 1.$

763. $\begin{cases} 2x'' - x' + 9x - y'' - y' - 3y = 0, \\ 2x'' + x' + 7x - y'' + y' - 5y = 0, \end{cases}$

$$x(0) = x'(0) = 1, \ y(0) = y'(0) = 0.$$

764. $\begin{cases} x' + y' - y = e^t, \\ 2x' + y' + 2y = \cos t, \end{cases} \quad x(0) = y(0) = 0.$

765. $\begin{cases} x' = -x + y + z + e^t, \\ y' = x - y + z + e^{3t}, \\ z' = x + y + z + 4, \end{cases}$ $x(0) = y(0) = z(0) = 0.$

766. $\begin{cases} x' = -y - z, \\ y' = -x - z, \\ z' = -x - y, \end{cases}$ $x(0) = -1,\ y(0) = 0,\ z(0) = 1.$

767. $\begin{cases} x' = y + z, \\ y' = 3x + z, \\ z' = 3x + y, \end{cases}$ $x(0) = 0,\ y(0) = 1,\ z(0) = 1.$

768. $\begin{cases} x' = 3y - x, \\ y' = y + x + e^{nt}, \end{cases}$ $x(0) = 1,\ y(0) = 1.$

769. $\begin{cases} x' = 2x - y + z, \\ y' = x + z, \\ z' = -3x + y - 2z, \end{cases}$ $x(0) = 1,\ y(0) = 1,\ z(0) = 0.$

770. $\begin{cases} x' = -2x - 2y - 4z, \\ y' = -2x + y - 2z, \\ z' = 5x + 2y + 7z, \end{cases}$ $x(0) = y(0) = z(0) = 1.$

771. $\begin{cases} tx' = -x + y + z + t, \\ ty' = x - y + z + t^3, \\ tz' = x + y + z + 4, \end{cases}$ $x(1) = y(1) = z(1) = 0.$

772. $\begin{cases} x_0' = -cx_0, \\ x_1' = -cx_1 + cx_0, \\ \quad \cdots\cdots\cdots\cdots \\ x_n' = -cx_n + cx_{n-1}, \end{cases}$

$x_0(0) = 1,\ x_1(0) = x_2(0) = \ldots = x_n(0) = 0.$

773. $\begin{cases} 3x' + 2x + y' = 1, \\ x' + 4y' + 3y = 0, \end{cases}$ $x(0) = y(0) = 0.$

774. $\begin{cases} 3tx' = 2x + y - z, \\ 2ty' = x + 3y + z, \\ 6tz' = -x + 7y + 5z, \end{cases}$ $x(1) = y(1) = z(1) = 1.$

775. $\begin{cases} x' - x - 2y = t, \\ -2x + y' - y = t, \end{cases}$ $x(0) = 2,\ y(0) = -4.$

776. An electron is emitted at the origin of coordinates O with an initial velocity $\mathbf{v}_0$ directed along the x axis. Find the equation of motion of the electron assuming that there is a constant magnetic field whose vector $\mathbf{H}$ is at right angles to the xOy plane.

777. A projectile is fired from a cannon with a muzzle velocity $\mathbf{v}_0$ m/s at an angle of 45° to the horizon. Neglecting drag, calculate the range and maximum height that the projectile reaches.

778. An electron is moving in a constant magnetic field $\mathbf{H}$. Find the electron trajectory if the initial velocity of the electron $\mathbf{v}_0$ forms an angle α with $\mathbf{H}$.

779. Find the equation of motion of a particle of mass m that is thrown with a velocity $\mathbf{v}_0$ at an angle α to the horizon in a medium that exerts a resisting force proportional to the first power of the particle's velocity, $\mathbf{F} = mk\mathbf{v}$.

780. A particle of mass m and electric charge e is ejected at the origin of coordinates O with a velocity $(u, 0, 0)$. It moves in a medium that exerts a resisting force $km\mathbf{v}$, where $\mathbf{v}$ is the particle velocity, in addition to a constant magnetic field $\mathbf{H}$ directed along the z axis. Show that the coordinates of the particle at time $t \neq 0$ are

$$x = \frac{kue^{-kt}}{k^2+\lambda^2}\left(e^{kt} - \cos\lambda t + \frac{\lambda}{k}\sin\lambda t\right),$$

$$y = -\frac{\lambda u}{k^2+\lambda^2} + \frac{ue^{-kt}}{k^2+\lambda^2}(\lambda\cos\lambda t + k\sin\lambda t),$$

where $\lambda = eH/mc$, and c is the velocity of light in vacuum.

781. A particle is moving in a medium that exerts a resisting force $\mathbf{F} = 2\lambda\mathbf{v}$, where $\mathbf{v}$ is the particle velocity, and at the same time is attracted by the point (0, 0) with a force $\mu^2 r$ $(m = 1)$. At point $(a, 0)$ the velocity is v_0 and parallel to the y axis. Show that at $\mu > \lambda$ the trajectory of the particle is described by the equations

$$x = ae^{-\lambda t}\left[\cos\omega t + \frac{\lambda}{\omega}\sin\omega t\right],$$

$$y = \frac{v_0}{\omega}e^{-\lambda t}\sin\omega t,$$

where $\omega = \sqrt{\mu^2 - \lambda^2}$, and r is the distance from the moving particle to point (0, 0).

782. A point A of mass m rests at a distance a from the x axis. A velocity v_0 parallel to this axis is imparted to the point. At the same time the point is attracted to the x axis by a force F that is proportional to the distance from the x axis (the proportionality factor is mk^2). Find the equation of motion of the point and its trajectory.

§ 18 Solving Volterra's Integral Equations with Kernels of a Special Type

An *integral equation* is an equation that contains the unknown function under the integral sign. It is known, for instance, that the solution of the Cauchy problem

$$y' = f(x, y), \quad y(x_0) = y_0.$$

can be reduced to solving the following integral equation:

$$y = \int_{x_0}^{x} f(x, y(x))\, dx + y_0.$$

If the unknown function enters the equation linearly, the integral equation is said to be *linear*.

The equation of the type

$$y(x) = f(x) + \int_a^b K(x, t)\, y(t)\, dt. \tag{18.1}$$

(a and b are constant) is said to be a *nonhomogeneous (linear) Fredholm integral equation of the second kind*. Here $K(x, t)$ and $f(x)$ are given functions and $y(x)$ is the unknown function; $K(x, t)$ is called the *kernel* of Eq. (18.1).

The equation

$$y(x) = f(x) + \int_a^x K(x, t)\, y(t)\, dt \tag{18.2}$$

is said to be a *nonhomogeneous (linear) Volterra integral equation of the second kind*.

If in Eqs. (18.1) and (18.2) $f(x) \equiv 0$, the equations are said to be *homogeneous*.

If the unknown function is only under the integral sign, we have *Fredholm and Volterra equations of the first kind*, respectively:

$$\int_a^b K(x, t)\, y(t)\, dt = f(x) \text{ and } \int_a^x K(x, t)\, y(t)\, dt = f(x).$$

The equation of the type

$$\varphi(x) + \int_0^x K(x-t)\, \varphi(t)\, dt = f(x) \tag{18.3}$$

with a kernel $K(x-t)$, i.e. which depends only on the difference between the arguments, constitutes an important class of Volterra's equations, sometimes called *integral equations of convolution form.*

Suppose we have a Volterra integral equation of convolution form of the second kind:

$$\varphi(x) = f(x) + \int_0^x K(x-t)\, \varphi(t)\, dt. \tag{18.4}$$

We will assume that $f(x)$ and $K(x)$ are smooth functions and have a finite growth index for $x \geqslant 0$. In this case $\varphi(x)$ has a finite growth index for $x \geqslant 0$, too, which means we can find the result functions (the Laplace transforms) of f, K, and φ. We introduce $\Phi(p) \doteq \varphi(x)$, $F(p) \doteq f(x)$, and $L(p) \doteq K(x)$. Applying the Laplace transformation to both sides of Eq. (18.4) and using the convolution formula (see Property IX in § 14), we obtain

$$\Phi(p) = F(p) + L(p)\, \Phi(p), \tag{18.5}$$

whence

$$\Phi(p) = \frac{F(p)}{1 - L(p)}, \quad L(p) \neq 1. \tag{18.6}$$

We can then find the object function $\varphi(x)$, which is the solution of the integral equation (18.4).

Example 1. Solve the integral equation

$$\varphi(x) = \cos x + \int_0^x (x-t)\, \varphi(t)\, dt. \tag{18.7}$$

Solution. Going over to result functions and taking the integral as a convolution of two functions, we obtain (on the basis of the rule of finding the result function of a convolution)

$$\Phi(p) = \frac{p}{p^2+1} + \frac{1}{p^2}\Phi(p), \tag{18.8}$$

whence

$$\Phi(p) = \frac{p^3}{(p^2+1)(p^2-1)}. \tag{18.9}$$

If we now find the object function for $\Phi(p)$, we arrive at the solution of the integral equation (18.7):

$$\varphi(x) = \frac{1}{2}(\cos x + \cosh x). \tag{18.10}$$

Solve the following integral equations:

783. $\varphi(x) = \sin x + \int_0^x (x-t)\varphi(t)\,dt.$

784. $\varphi(x) = x + \frac{1}{2}\int_0^x (x-t)^2\varphi(t)\,dt.$

785. $\varphi(x) = x + \int_0^x \sin(x-t)\varphi(t)\,dt.$

786. $\varphi(x) = \cos x + \int_0^x e^{x-t}\varphi(t)\,dt.$

787. $\varphi(x) = 1 + x + \int_0^x \cos(x-t)\varphi(t)\,dt.$

788. $\varphi(x) = \frac{x^2}{2} + \int_0^x (x-t)\,e^{-(t-x)}\varphi(t)\,dt.$

789. $\varphi(x) = e^{-x} + \frac{1}{2}\int_0^x (x-t)^2\varphi(t)\,dt.$

790. $\varphi(x) = x + 2\int_0^x [(x-t) - \sin(x-t)]\,\varphi(t)\,dt.$

791. $\varphi(x) = \sin x + 2\int_0^x \cos(x-t)\,\varphi(t)\,dt.$

792. $\varphi(x) = 1 - 2x - 4x^2 + \int_0^x [3 + 6(x-t) - 4(x-t)^2]\,\varphi(t)\,dt.$

793. $\varphi(x) = 1 + \frac{1}{2}\int_0^x \sin 2(x-t)\,\varphi(t)\,dt.$

794. $\varphi(x) = e^x - 2\int_0^x \cos(x-t)\,\varphi(t)\,dt.$

795. $\varphi(x) = 1 + \frac{1}{6}\int_0^x (x-t)^3\,\varphi(t)\,dt.$

796. $\varphi(x) = x - \int_0^x \sinh(x-t)\,\varphi(t)\,dt.$

797. $\varphi(x) = \sinh x - \int_0^x \cosh(x-t)\,\varphi(t)\,dt.$

We can use the same method to solve Volterra's integral equations of the first kind with kernels $K(x, t)$ that depend only on the difference $x - t$, i.e. equations of the convolution form

$$\int_0^x K(x-t)\,\varphi(t)\,dt = f(x), \qquad (18.11)$$

where $f(x)$ is known and $\varphi(x)$ unknown. (We assume that $K(x, x) \neq 0$.) Suppose $F(p) \doteqdot f(x)$, $L(p) \doteqdot K(x)$, and $\Phi(p) \doteqdot \varphi(x)$. Applying the Laplace transformation to both sides of Eq. (18.11) and using the property of convolution of object functions, we obtain

$$L(p)\,\Phi(p) = F(p),$$

whence

$$\Phi(p) = \frac{F(p)}{L(p)}, \quad L(p) \neq 0.$$

The object function for $\Phi(p)$ is the solution $\varphi(x)$ of Eq. (18.11).

Solve the following integral equations:

798. $\int_0^x e^{x-t}\varphi(t)\,dt = x.$

799. $\int_0^x J_0(x-t)\,\varphi(t)\,dt = \sin x.$

800. $\int_0^x \cos(x-t)\,\varphi(t)\,dt = \sin x.$

801. $\int_0^x e^{x-t}\varphi(t)\,dt = \sin x.$

802. $\int_0^x \cos(x-t)\,\varphi(t)\,dt = x + x^2.$

803. $\int_0^x e^{2(x-t)}\varphi(t)\,dt = x^2 e^x.$

804. $\int_0^x \cosh(x-t)\,\varphi(t)\,dt = \sinh x.$

805. $\int_0^x \cosh(x-t)\,\varphi(t)\,dt = x.$

The above method of solving equations of the convolution form (18.4) and (18.11) can also be applied to systems of Volterra-type integral equations of the convolution form:

$$\varphi_i(x) = f_i(x) + \sum_{k=1}^{s} \int_0^x K_{ik}(x-t)\,\varphi_k(t)\,dt \quad (i = 1, 2, \ldots, s). \tag{18.12}$$

Applying the Laplace transformation to both sides of (18.12), we obtain

$$\Phi_i(p) = F_i(p) + \sum_{k=1}^{s} L_{ik}(p)\,\Phi_k(p) \quad (i = 1, 2, \ldots, s).$$

If we solve this system of equations, which is linear in $\Phi_i(p)$, we will find the $\Phi_i(p)$ $(i = 1, 2, \ldots, s)$, whose object functions give the solution of the system of integral equations (18.12).

Example 2. Solve the system of integral equations

$$\begin{cases} \varphi_1(x) = x + \int_0^x e^{-(x-t)}\varphi_1(t)\,dt + \int_0^x (x-t)\,\varphi_2(t)\,dt, \\ \varphi_2(x) = 1 + \int_0^x \sinh(x-t)\,\varphi_1(t)\,dt - \int_0^x e^{(x-t)}\varphi_2(t)\,dt. \end{cases}$$

Solution. Going over to result functions and using the property of convolution of object functions, we obtain ($\Phi_1(p) \doteq \varphi_1(x)$ and $\Phi_2(p) \doteq \varphi_2(x)$)

$$\begin{cases} \Phi_1(p) = \frac{1}{p^2} + \frac{1}{p+1}\Phi_1(p) + \frac{1}{p^2}\Phi_2(p), \\ \Phi_2(p) = \frac{1}{p} + \frac{1}{p^2-1}\Phi_1(p) - \frac{1}{p-1}\Phi_2(p), \end{cases}$$

whence

$$\Phi_1(p) = \frac{p^2+p-1}{p(p-1)(p^2+1)}, \quad \Phi_2(p) = \frac{p^3-p^2+1}{(p-1)(p+1)(p^2+1)}.$$

The object functions for $\Phi_1(p)$ and $\Phi_2(p)$ are

$$\varphi_1(x) = 1 + \frac{1}{2}e^x + \frac{1}{2}\sin x - \frac{3}{2}\cos x,$$

$$\varphi_2(x) = \frac{1}{2}(\cos x + \cosh x) - \sin x.$$

The system of functions $\varphi_1(x)$ and $\varphi_2(x)$ constitutes the solution of the problem.

Solve the following systems of integral equations:

806. $$\begin{cases} \varphi_1(x) = 1 - 2\int_0^x e^{2(x-t)}\varphi_1(t)\,dt + \int_0^x \varphi_2(t)\,dt, \\ \varphi_2(x) = 4x - \int_0^x \varphi_1(t)\,dt + 4\int_0^x (x-t)\,\varphi_2(t)\,dt. \end{cases}$$

807. $$\begin{cases} \varphi_1(x) = e^x + \int_0^x \varphi_1(t)\,dt - \int_0^x e^{(x-t)}\varphi_2(t)\,dt, \\ \varphi_2(x) = -x - \int_0^x (x-t)\,\varphi_1(t)\,dt + \int_0^x \varphi_2(t)\,dt. \end{cases}$$

808. $$\begin{cases} \varphi_1(x) = x + \int_0^x \varphi_1(t)\,dt + \int_0^x (x-t)\,\varphi_2(t)\,dt, \\ \varphi_2(x) = 1 - \int_0^x e^{(x-t)}\varphi_1(t)\,dt + \int_0^x \varphi_2(t)\,dt. \end{cases}$$

809. $$\begin{cases} \varphi_1(x) = e^x - \int_0^x \varphi_1(t)\,dt + 4\int_0^x e^{(x-t)}\varphi_2(t)\,dt, \\ \varphi_2(x) = 1 - \int_0^x e^{-(x-t)}\varphi_1(t)\,dt + \int_0^x \varphi_2(t)\,dt. \end{cases}$$

810. $$\begin{cases} \varphi_1(x) = 2x - \int_0^x (x-t)\,\varphi_1(t)\,dt + \int_0^x \varphi_2(t)\,dt, \\ \varphi_2(x) = -2 - 4\int_0^x \varphi_1(t)\,dt + 3\int_0^x (x-t)\,\varphi_2(t)\,dt. \end{cases}$$

811. $$\begin{cases} \varphi_1(x) = 2 - \int_0^x (x-t)\,\varphi_1(t)\,dt - 4\int_0^x \varphi_2(t)\,dt, \\ \varphi_2(x) = 1 - \int_0^x \varphi_1(t)\,dt - \int_0^x (x-t)\,\varphi_2(t)\,dt. \end{cases}$$

§ 19 Differential Equations with Retarded Arguments

In a number of technical problems one has to deal with differential equations in which the unknown function enters with different values of its arguments, for instance,

$$\dot{x}(t) = \varphi(t, x(t), x(t - \tau(t))), \tag{19.1}$$

$$\dot{x}(t) = \varphi(t, x(t), x(t - \tau(t)), \dot{x}(t - \tau(t))), \tag{19.2}$$

$$\ddot{x}(t) = \varphi(t, x(t), \dot{x}(t), x(t - \tau_1(t)), \dot{x}(t - \tau_2(t))). \tag{19.3}$$

Such equations are called *differential equations with deviating arguments*. If the $\tau_i(t)$ are constant, then we call these equations *difference-differential equations*. Finally, if the τ_i are positive and the higher derivative enters a difference-differential equation only with one value of its argument that is not less than the values of other arguments of the functions and their derivatives in the equation, the equation is called a *differential equation with a retarded argument*.

Suppose we have a differential equation with a retarded argument and constant coefficients:

$$x^{(n)}(t) = \sum_{k=0}^{n-1} a_k x^{(k)}(t - \tau_k) + f(t), \tag{19.4}$$

where the a_k are constants and the τ_k are nonnegative constants ($0 < t < +\infty$). For the sake of simplicity we take zero initial conditions,

$$x(0) = x'(0) = \ldots = x^{(n-1)}(0) = 0, \tag{19.5}$$

and assume that

$$x(t) = x'(t) = \ldots = x^{(n-1)}(t) \equiv 0 \text{ for } t < 0.$$

Applying the Laplace transformation to both sides of (19.4) and using the property of translation of object function (Property VIII in § 14), we arrive at an operator equation for $X(p) \doteqdot x(t)$:

$$p^n X(p) = \sum_{k=0}^{n-1} a_k p^k X(p) e^{-\tau_k p} + F(p),$$

$$\text{where } F(p) \doteqdot f(t), \tag{19.6}$$

whence

$$X(p)=\frac{F(p)}{p^n-\sum\limits_{k=0}^{n-1}a_k p^k e^{-\tau_k p}}. \tag{19.7}$$

The object function for $X(p)$ determined via (19.7) gives the solution $x(t)$ of Eq. (19.4) that satisfies the initial conditions (19.5).

Example 1. Solve the equation

$$x'(t)=x(t-1)+1, \quad x(0)=0.$$

Solution. Going over to result functions, we obtain

$$pX(p)=X(p)e^{-p}+\frac{1}{p},$$

whence

$$X(p)=\frac{1}{p}\frac{1}{p-e^{-p}}=\frac{1}{p^2}\frac{1}{1-\frac{e^{-p}}{p}}$$

$$=\frac{1}{p^2}\left(1+\frac{e^{-p}}{p}+\frac{e^{-2p}}{p^2}+\ldots+\frac{e^{-np}}{p^n}+\ldots\right).$$

Going back to the object function for $x(t)$, we obtain

$$x(t)=t\eta(t)+\frac{1}{2!}(t-1)^2\eta(t-1)+\ldots$$

$$+\frac{1}{(n+1)!}(t-n)^{n+1}\eta(t-n)+\ldots$$

$$=\sum_{k=0}^{\infty}\frac{(t-k)^{k+1}}{(k+1)!}\eta(t-k).$$

Solve the following equations:

812. $x''(t)-x(t-1)=t$, $x(0)=x'(0)=0$.

813. $x''(t)-2x'(t-1)=t$, $x(0)=x'(0)=0$.

814. $x''(t)=2x'(t-1)-x(t-2)+1$, $x(0)=x'(0)=0$.

815. $x''(t)+2x'(t-2)+x(t-4)=t$, $x(0)=x'(0)=0$.

For differential equations with a retarded argument, which describe processes with aftereffect, we often encounter problems of the following type:

Find the solution $x(t)$ for $t\geqslant t_0$, while for all $t\leqslant t_0$ for which the values of $x(t)$ affect the subsequent values of the solution at $t\geqslant 0$ the function $x(t)$ is known.

For instance, one such problem goes as follows: Find the continuous solution $x(t)$ at $t \geqslant t_0$ of the equation

$$x(t) = f(t, x(t), x(t-\tau)), \quad \tau > 0 = \text{const},$$

if we know that $x(t) = \varphi(t)$ for $t_0 - \tau \leqslant t \leqslant t_0$. Here $\varphi(t)$ is a given continuous function, known as the *initial function*, and the segment $[t_0 - \tau, t_0]$ where the function $\varphi(t)$ is defined as the *initial set*.

When the initial function is not identically zero, we can still use the Laplace transformation to find the solution of the linear equation (19.4) with constant coefficients and constant retardation. Here is an example.

Example 2. Solve the equation

$$x'(t) = x(t-1), \quad \varphi(t) \equiv 1, \quad -1 \leqslant t \leqslant 0.$$

Solution.

$$x(t) \doteqdot X(p), \quad x'(t) \doteqdot pX(p) - x(0) = pX(p) - 1.$$

Applying the Laplace transform to both sides of the equation, we obtain

$$pX(p) - 1 = \int_0^\infty e^{-pt} x(t-1)\, dt.$$

The change of variables $t - 1 = z$ yields

$$\begin{aligned}\int_0^\infty e^{-pt} x(t-1)\, dt &= \int_{-1}^\infty e^{-p(z+1)} x(z)\, dz \\ &= e^{-p} \int_{-1}^0 e^{-pz} x(z)\, dz + e^{-p} \int_0^\infty e^{-pz} x(z)\, dz \\ &= e^{-p} \frac{e^{-pz}}{-p}\Big|_{z=-1}^{z=0} + e^{-p} X(p),\end{aligned}$$

since $x(z) \equiv 1$ for $-1 \leqslant z \leqslant 0$. Finally,

$$pX(p) - 1 = \frac{1 - e^{-p}}{p} + e^{-p} X(p).$$

From this it follows that

$$X(p) = \frac{1}{p-e^{-p}} + \frac{1-e^{-p}}{p(p-e^{-p})} = \frac{1}{p\left(1-\frac{e^{-p}}{p}\right)} + \frac{1-e^{-p}}{p^2\left(1-\frac{e^{-p}}{p}\right)}$$

$$= \frac{1}{p}\left(1+\frac{e^{-p}}{p}+\frac{e^{-2p}}{p^2}+\ldots+\frac{e^{-kp}}{p^k}+\ldots\right)$$

$$+\frac{(1-e^{-p})}{p^2}\left(1+\frac{e^{-p}}{p}+\ldots+\frac{e^{-kp}}{p^k}+\ldots\right)$$

$$\frac{1}{p}+\frac{1}{p^2}+\frac{e^{-p}}{p^3}+\ldots+\frac{e^{-kp}}{p^{k+2}}+\ldots.$$

If we now find the object function for $X(p)$, we have the solution of the equation:

$$x(t) = (1+t)\,\eta(t) + \sum_{k=2}^{\infty} \frac{(t-k+1)^k}{k!}\,\eta(t-k+1).$$

Solve the following equations:

816. $x'(t) = x(t-1)$, $\varphi(t) = t$, $-1 \leqslant t \leqslant 0$.

817. $x'(t) = x(t-1) + t$, $\varphi(t) \equiv 1$, $-1 \leqslant t \leqslant 0$.

818. $x'(t) + x\left(t-\frac{\pi}{2}\right) = 0$, $\varphi(t) = \cos t$, $-\frac{\pi}{2} \leqslant t \leqslant 0$.

§ 20 Solving Some Problems of Mathematical Physics

We will consider the case where the unknown function u depends only on two independent variables, x and t. The variable x is taken as a spatial variable while t as a time variable.

We start with the *heat-conduction equation*

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,\ t) \tag{20.1}$$

(a^2 is constant). Let us take the first boundary-value problem for Eq. (20.1): To find the solution $u(x,\ t)$ of the differential equation (19.1) for $0 \leqslant x \leqslant l$ and $t \geqslant 0$ that satisfies the initial condition

$$u(x,\ 0) = \varphi(x) \tag{20.2}$$

and the boundary conditions

$$u(0,\ t) = \psi_1(t),\quad u(l,\ t) = \psi_2(t). \tag{20.3}$$

Let us assume that $u(x, t)$, $\partial^2 u(x, t)/\partial x^2$, and $f(x, t)$, if considered as functions of t, are object functions. We introduce

$$U(p, x) = \int_0^\infty u(x, t)\, e^{-pt}\, dt \tag{20.4}$$

as the result function of $u(x, t)$. Then

$$\frac{\partial u}{\partial x} \doteqdot \int_0^\infty \frac{\partial u}{\partial x} e^{-pt}\, dt = \frac{dU}{dx}, \quad \frac{\partial^2 u}{\partial x^2} \doteqdot \frac{d^2U}{dx^2}. \tag{20.5}$$

By the property of differentiation of object functions and bearing in mind the initial condition (20.2), we obtain

$$\frac{\partial u}{\partial t} \doteqdot pU - \varphi(x). \tag{20.6}$$

Now we assume that $\psi_1(t)$ and $\psi_2(t)$ are object functions and

$$\psi_1(t) \doteqdot \Psi_1(p), \quad \psi_2(t) \doteqdot \Psi_2(p). \tag{20.7}$$

Then the boundary conditions (20.3) yield

$$U|_{x=0} = \Psi_1(p), \quad U|_{x=l} = \Psi_2(p). \tag{20.8}$$

In this way we have used the operational method to reduce the boundary-value problem (20.1)-(20.3) to an ordinary differential equation

$$a^2 \frac{d^2U}{dx^2} - pU + \varphi(x) + F(x, p) = 0 \tag{20.9}$$

with the boundary conditions (20.8), where $F(x, p) \doteqdot f(x, t)$. Solving the problem (20.9), (20.8) and going back to object functions, we arrive at the solution $u(x, t)$ of the boundary-value problem (20.1)-(20.3). Other boundary-value problems for the heat-conduction equation, and boundary-value problems for the *equation of string vibrations*

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t), \tag{20.10}$$

the *telegrapher's equation* (see *Barbashin*, and *Kruchkovich et al.*)

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{e^2 u}{\partial x^2} + (\alpha + \beta) \frac{\partial u}{\partial t} + \alpha\beta u = 0, \tag{20.11}$$

and some other equations of a more general type are solved along the same lines.

Example. The ends of a string are fixed at $x = 0$ and $x = l$. The initial deflection is given by the equation $u(x, 0) = A \sin(\pi x/l)$, where $0 \leqslant x \leqslant l$. Finally, the initial velocities are zero. Find the deflection $u(x, t)$ at $t > 0$.

Solution. What we must do is solve the differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{20.12}$$

with the initial conditions

$$u(x, 0) = A \sin\frac{\pi x}{l}, \qquad \frac{\partial u(x, 0)}{\partial t} = 0, \tag{20.13}$$

and the boundary conditions

$$u(0, t) = u(l, t) = 0. \tag{20.14}$$

Going over to result functions, we obtain

$$\frac{d^2U}{dx^2} - \frac{p^2}{a^2}U = -\frac{pA}{a^2}\sin\frac{\pi x}{l}, \tag{20.15}$$

$$U|_{x=0} = U|_{x=l} = 0. \tag{20.16}$$

Solution of Eq. (20.15) yields

$$U(x, p) = C_1 e^{px/a} + C_2 e^{-px/a} + \frac{Ap}{p^2 + \frac{a^2\pi^2}{l^2}}\sin\frac{\pi x}{l}.$$

Taking into account the boundary-value conditions (20.16), we find that

$$U(x, p) = \frac{Ap}{p^2 + \frac{a^2\pi^2}{l^2}}\sin\frac{\pi x}{l}.$$

The object function for $U(x, p)$ is

$$u(x, t) = A\cos\frac{\pi a t}{l}\sin\frac{\pi x}{l},$$

which constitutes the solution of the problem.

Solve the following problems:

819. $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ $(x > 0,\ t > 0)$, $u(0, t) = u_0$, $u(x, 0) = 0$.

820. $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ $(x > 0,\ t > 0)$, $u(0, t) = 0$,

$u(x, 0) = u_1$.

821. $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ $(x > 0, t > 0)$,

$u(0, t) = a \cos \omega t$, $u(x, 0) = 0$.

822. $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ $(x > 0, t > 0)$,

$u(0, t) = a \sin \omega t$, $u(x, 0) = 0$.

823. $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ $(x > 0, t > 0)$,

$u(0, t) = \varphi(t)$, $u(x, 0) = 0$.

824. Find the temperature distribution in a rod $0 \leqslant x \leqslant l$ under the condition that there is no heat flux through the end $x = 0$; the other end $x = l$ is kept at a constant temperature u_1, and the initial temperature of the rod is $u_0 = \text{const}$.

825. Find the temperature distribution in a semi-infinite rod whose left end gives off heat to a medium with a zero temperature. The initial temperature of the rod is $u_0 = \text{const}$.

826. A rod of length l is in the state of rest and its end $x = 0$ is secured. At time zero a force F (per unit rod area) is applied to the rod; F is directed along the rod. Find the law describing the vibrations of the rod.

827. A rod hangs vertically and is secured in such a way that the diflection at all points is zero. At time zero the lower end is released while the upper end remains secured. Find the law of vibrations of the rod.

828. Solve the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + bx(x - l)$$

with zero initial conditions and the following boundary conditions:

$$u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad u(0, t) = u(l, t) = 0.$$

829. A homogeneous string, kept fixed at its ends $x = 0$ and $x = l$, has at time zero the shape of a parabola symmetric with respect to the straight line that is at right angles

to the string at point $x = l/2$. Find the deflections of the string from the rectilinear position of equilibrium, assuming that initial velocities are zero.

§ 21 Discrete Laplace Transformation

Suppose we have a complex-valued function $f(t)$ of a real argument t, and the function is defined for $t \geqslant 0$. Let us take the sequence $\{f(n)\}$ $(n = 0, 1, 2, \ldots)$, which we will denote by $f(n)$ and call the *lattice function*; $f(t)$ is said to be the *generating function of* $f(n)$. Thus, the argument of a lattice function takes on only integral values, and for negative values of the argument the lattice function is zero.

The *discrete Laplace transform* of a lattice function $f(n)$ is the function $F^*(p)$ of a complex variable $p = s + i\sigma$ defined thus:

$$F^*(p) = \sum_{n=0}^{\infty} e^{-np} f(n); \tag{21.1}$$

we assume that the series on the right-hand side is convergent.

We call $f(n)$ the *object function* and $F^*(p)$ its *result function* and write

$$F^*(p) \doteq f(n) \quad \text{or} \quad f(n) \doteq F^*(p).$$

The transformation from $f(n)$ to $F^*(p)$ via (21.1) is called the *discrete Laplace transformation.*

The value s^* of $\operatorname{Re} p$ for which the series in (21.1) is convergent for $\operatorname{Re} p = s > s^*$ and divergent for $s < s^*$ is called the *abscissa of convergence.* The function $F^*(p)$ is periodic with a period $2\pi i$ and analytic in the half-plane $\operatorname{Re} p > s^*$.

If a lattice function $f(n)$ satisfies the condition that

$$|f(n)| \leqslant Me^{\lambda_0 n}, \tag{21.2}$$

then the abscissa of convergence s^* is greater than λ_0 and, hence, $f(n)$ has a result function. In general, each function $f(t)$ that is the object function for the common Laplace transform generates a lattice function $f(n)$ which has a discrete Laplace transform $F^*(p)$.

Example 1. Using the above definitions, find the result function of

$$f(n) = e^{-n}.$$

Solution. Obviously, this function satisfies condition (21.2) with $\lambda_0 = 1$ and an arbitrary $M > 1$. This means that it has a result function. According to (21.1),

$$F^*(p) = \sum_{n=0}^{\infty} e^{-pn} e^{-n} = \sum_{n=0}^{\infty} e^{-(1+p)n}$$

$$= \frac{1}{1 - e^{-(1+p)}} \quad (\operatorname{Re} p > -1). \tag{21.3}$$

Note that the lattice function $f(n) = e^{n^2}$ has no result function, since its abscissa of convergence s^* is infinite.

Using the above definitions, find result functions of the following functions:

830. $f(n) = \eta(n)$, where $\eta(n) = \begin{cases} 1 \text{ for } n > 0, \\ 0 \text{ for } n < 0. \end{cases}$

831. $f(n) = n$.

832. $f(n) = e^{an}$.

833. $f(n) = a^n \quad (a > 0,\ a \neq 1)$.

Properties of the discrete Laplace transformation

I. *Linearity.* For any complex constants α and β,

$$\alpha f(n) + \beta g(n) \doteq \alpha F^*(p) + \beta G^*(p).$$

(Here and everywhere below we assume that $f(n) \doteq F^*(p)$ and $g(n) \doteq G^*(p)$.)

Example 2. Find the result function of $f(n) = \sin n$.

Solution. By Euler's formulas,

$$\sin n = \frac{e^{in} - e^{-in}}{2i} = \frac{1}{2i} e^{in} - \frac{1}{2i} e^{-in}.$$

We have

$$e^{in} \doteq \sum_{n=0}^{\infty} e^{-np} e^{in} = \frac{1}{1 - e^{-(p-i)}},$$

$$e^{-in} \doteq \frac{1}{1 - e^{-(p+i)}}.$$

By the property of linearity,

$$\sin n \doteq \frac{1}{2i}\left(\frac{1}{1-e^{-(p-i)}}-\frac{1}{1-e^{-(p+i)}}\right) = \frac{e^p \sin 1}{e^{2p}-2e^p\cos 1+1}.$$

Find the result functions of the following functions:

834. $f(n) = \cos n$.

835. $f(n) = \sin \alpha n$ ($\alpha = \text{const}$).

836. $f(n) = \sinh n$.

837. $f(n) = e^n - 2e^{n/2}$.

838. $f(n) = \cos^2 n$.

II. *Translation of object function.* We assume that $f(n) \doteq F^*(p)$ and k is a positive integer. Then

$$f(n+k) \doteq e^{kp}\left[F^*(p) - \sum_{m=0}^{k-1} e^{-mp} f(m)\right]. \qquad (21.4)$$

For one, if $f(0) = f(1) = \ldots = f(k-1) = 0$, then

$$f(n+k) \doteq e^{kp} F^*(p). \qquad (21.5)$$

Similarly,

$$f(n-k) \doteq e^{-kp} F^*(p) \quad (f(n-k) \equiv 0 \text{ for } n < k). \qquad (21.6)$$

Example 3. Find the result function of $f(n) = e^{n-2}$ ($f(n) \equiv 0$ for $n < 2$).

Solution. We have

$$e^n \doteq \frac{1}{1-e^{1-p}} = \frac{ep}{ep-e}.$$

By the property of translation of object function (see (21.6)),

$$e^{n-2} \doteq e^{-2p}\frac{ep}{ep-e} = \frac{1}{ep(ep-e)}.$$

Find the result functions of the following functions:

839. $f(n) = \eta(n-k)$. 840. $f(n) = e^{\alpha(n+3)}$.

841. $f(n) = \sinh 2(n-1) \times \eta(n-1)$.

842. $f(n) = (n+2)^2$.

III. *Translation of result function.* If $F^*(p) \doteq f(n)$, then for any complex-valued p_0 we have

$$F^*(p-p_0) \doteq e^{p_0 n} f(n). \qquad (21.7)$$

Example 4. Find the result function of $f(n) = ne^{2n}$.

Solution. We have

$$n \doteq \frac{e^{-p}}{(1-e^{-p})^2}.$$

By the property of translation of result function ($p_0 = 2$),

$$e^{2n} n \doteq \frac{e^{-(p-2)}}{[1-e^{-(p-2)}]^2}.$$

Find the result functions of the following functions:

843. $f(n) = e^{-p} \sin 2n$.

844. $f(n) = n^2 e^{2n}$.

845. $f(n) = e^{3n} \cosh n$.

IV. *Differentiation of result function.* This is done by multiplying the corresponding object function by $-n$:

$$\frac{d}{dp}\{F^*(p)\} \doteq -nf(n), \tag{21.8}$$

and in general

$$\frac{d^k}{dp^k}\{F^*(p)\} \doteq (-1)^k n^k f(n). \tag{21.9}$$

Example 5. Find the result function of $f(n) = ne^n$.

Solution. We have

$$e^n \doteq \frac{e^p}{e^p - e}.$$

By the property of differentiation of result function,

$$ne^n \doteq -\frac{d}{dp}\left(\frac{e^p}{e^p - e}\right) = \frac{e^{p+1}}{(e^p - e)^2}.$$

Find the result functions of the following functions:

846. $f(n) = n^2 e^n$.

847. $f(n) = n^2$.

848. $f(n) = n \sin\left(n \frac{\pi}{2}\right)$.

V. *Integration of result function.* Let the lattice function $f(n)$ satisfy the following conditions:

$$f(0) = 0, \quad \left.\frac{f(t)}{t}\right|_{t=0} = \lim_{t \to +0} \frac{f(t)}{t} = 0. \tag{21.10}$$

Then

$$\frac{f(n)}{n} \doteq \int_p^\infty F^*(p)\, dp, \tag{21.11}$$

i.e. division of the object function by n is equivalent to integration of the result function from p to ∞.

Remarks. (a) At $f(0) \neq 0$ the integral on the right-hand side of (21.11) is divergent and, hence, the property of integration of a result function will not hold.

(b) If

$$\frac{f(t)}{t}\Big|_{t=0} = \lim_{t\to+0}\frac{f(t)}{t} = a \neq 0,$$

then

$$\frac{f(n)}{n} \doteq a + \int_p^\infty F^*(p)\,dp. \tag{21.12}$$

(c) If for $m = 1, 2, \ldots, k$ the following conditions are met:

$$\lim_{t\to+0}\frac{f(t)}{t^m} = 0,$$

then

$$\frac{f(n)}{n^k} \doteq \underbrace{\int_p^\infty \ldots \int_p^\infty}_{k} F^*(p)\,dp \ldots dp, \tag{21.13}$$

i.e. division of the object function by n^k is equivalent to integration of the result function k times from p to ∞.

Example 6. Find the result function of the lattice function $(e^n - 1 - n)/n$.

Solution. Let us assume that $f(n) = e^n - 1 - n$. We check whether conditions (21.10) are met. We have

$$f(0) = 0,$$

$$\lim_{t\to+0}\frac{f(t)}{t} = \lim_{t\to+0}\frac{e^t-1-t}{t} = \lim_{t\to+0}\left(\frac{e^t-1}{t} - 1\right) = 1 - 1 = 0.$$

Next we find the result function of $f(n)$:

$$e^n - 1 - n \doteq \frac{e^p}{e^p-e} - \frac{e^p}{e^p-1} - \frac{e^p}{(e^p-1)^2}.$$

Since conditions (21.10) are met, we can write

$$\frac{e^n-1-n}{n} \doteq \int_p^\infty \left[\frac{e^p}{e^p-e} - \frac{e^p}{e^p-1} - \frac{e^p}{(e^p-1)^2}\right] dp$$

$$= \left[\ln(e^p-e) - \ln(e^p-1) + \frac{1}{e^p+1}\right]\Big|_p^\infty$$

$$= \left[\ln\frac{e^p-e}{e^p-1} + \frac{1}{e^p-1}\right]\Big|_p^\infty = \ln\frac{e^p-1}{e^p-e} - \frac{1}{e^p-1}.$$

Example 7. Find the result function of the lattice function $(\sinh \alpha n)/n$, with $\alpha \neq 0$.

Solution. Let us assume that $f(n) = \sinh \alpha n$. We have

$$f(0) = 0; \quad \left.\frac{f(t)}{t}\right|_{t=0} = \lim_{t \to +0} \frac{\sinh \alpha t}{t} = \alpha \neq 0.$$

Next we find the result function of $f(n)$:

$$\sinh \alpha n \doteq \frac{1}{2}\left(\frac{e^p}{e^p - e^{\alpha}} - \frac{e^p}{e^p - e^{-\alpha}}\right).$$

Using (21.12), we can find the result function of the given lattice function;

$$\frac{\sinh \alpha n}{n} \doteq \alpha + \frac{1}{2}\int_p^{\infty}\left(\frac{e^p}{e^p - e^{\alpha}} - \frac{e^p}{e^p - e^{-\alpha}}\right) dp$$

$$= \alpha + \frac{1}{2}\left[\ln (e^p - e^{\alpha}) - \ln (e^p - e^{-\alpha})\right]\Big|_p^{\infty}$$

$$= \alpha + \frac{1}{2}\ln \frac{e^p - e^{\alpha}}{e^p - e^{-\alpha}}\Big|_0^{\infty} = \alpha + \frac{1}{2}\ln \frac{e^p - e^{-\alpha}}{e^p - e^{\alpha}}.$$

Find the result functions of the following functions

849. $\dfrac{a^n - 1}{n}$ $(a > 0,\ a \neq 1)$. **850.** $\dfrac{\sin \alpha n}{n}$ $(\alpha \neq 0)$.

851. $\dfrac{1 - \cos \alpha n}{n}$. **852.** $\dfrac{n - \sinh n}{n}$.

VI. *Differentiation with respect to a parameter.* Suppose both the object function and its result function contain a parameter ε independent of n and p and

$$F^*(p, \varepsilon) \doteq f(n, \varepsilon).$$

Then

$$\frac{\partial F^*(p, \varepsilon)}{\partial \varepsilon} \doteq \frac{\partial f(n, \varepsilon)}{\partial \varepsilon}, \tag{21.14}$$

i.e. the derivative of the result function with respect to ε is the result function of the derivative of the object function with respect to ε. We assume that all these derivatives exist and that $\partial f(n, \varepsilon)/\partial \varepsilon$ is an object function.

Example 8. Find the result function of $ne^{\alpha n}$, with α real.

Solution. We have $e^{\alpha n} \doteq e^p/(e^p - e^{\alpha})$. We take α as the parameter. By the property of differentiation with respect

to : parameter,

$$\left(\frac{e^p}{e^p-e^\alpha}\right)'_\alpha \doteq (e^{\alpha n})'_\alpha = ne^{\alpha n}.$$

Whence

$$ne^{\alpha n} \doteq \frac{e^p e^\alpha}{(e^p-e^\alpha)^2}.$$

Find the result functions of the following functions:

853. $f(n) = n\cos \alpha n$. 854. $f(n) = n^2 \sinh \alpha n$.

855. $f(n) = (n+2)\cosh \alpha n$.

VII. *Integration with respect to a parameter.* If $f(n, \varepsilon) \doteq F^*(p, \varepsilon)$, where parameter ε is independent of n and p $(\varepsilon_0 \leqslant \varepsilon \leqslant \lambda)$, then

$$\int_{\varepsilon_0}^{\varepsilon} f(n, \varepsilon)\, d\varepsilon \doteq \int_{\varepsilon_0}^{\varepsilon} F^*(p, \varepsilon)\, d\varepsilon, \qquad (21.15)$$

i.e. the integral of the object function with respect to parameter ε is the object function for the integral of the result function with respect to ε.

Example 9. Using the property of integration with respect to a parameter, find the result function of the lattice function $(1-\cos \varepsilon n)/n$.

Solution. We have

$$\sin \varepsilon n \doteq \frac{e^p \sin \varepsilon}{e^{2p}-2e^p\cos \varepsilon+1}.$$

Integrating both sides with respect to ε from $\varepsilon_0 = 0$ to ε, we have

$$\int_0^{\varepsilon} \sin \varepsilon n\, d\varepsilon \doteq \int_0^{\varepsilon} \frac{e^p \sin \varepsilon\, d\varepsilon}{e^{2p}-2e^p\cos \varepsilon+1}.$$

This yields

$$\frac{1-\cos \varepsilon n}{n} \doteq \frac{1}{2}\ln (e^{2p}-2e^p\cos \varepsilon+1)\Big|_0^{\varepsilon}$$

$$= \frac{1}{2}[\ln (e^{2p}-2e^p\cos \varepsilon+1) - \ln (e^{2p}-2e^p+1)].$$

Thus,

$$\frac{1-\cos \varepsilon n}{n} \doteq \frac{1}{2}\ln \frac{e^{2p}-2e^p\cos \varepsilon+1}{(e^p-1)^2}.$$

Using the property of integration with respect to a parameter, find the result functions of the following functions:

856. $\dfrac{e^{\varepsilon n}-e^{\varepsilon_0 n}}{n}$. 857. $(\cosh \varepsilon n - \cosh n)/n$.

858. $\dfrac{\sin \varepsilon n}{n}$. 859. $\dfrac{\sin (\varepsilon-1)\, n \times \cos (\varepsilon+1)\, n}{n}$.

VIII. *Convolution of object lattice functions.* Suppose

$$f_1(n) \doteqdot F_1^*(p), \quad f_2(n) \doteqdot F_2^*(p).$$

Then

$$F_1^*(p)\, F_2^*(p) \doteqdot \sum_{m=0}^{n} f_1(n-m)\, f_2(m) = \sum_{m=0}^{n} f_1(m)\, f_2(n-m). \tag{21.16}$$

Example 10. Find the object function whose result function is

$$F^*(p) = \frac{e^{2p}}{(e^p-e)(e^p-e^{-1})}.$$

Solution. The result function $F^*(p)$ can be written in the form of the product of two result functions:

$$F_1^*(p) = \frac{e^p}{e^p-e} \doteqdot e^n, \quad F_2^*(p) = \frac{e^p}{e^p-e^{-1}} \doteqdot e^{-n}.$$

By the property of convolution of lattice object functions,

$$F^*(p) \doteqdot \sum_{m=0}^{n} e^{-m} e^{n-m} = e^n \sum_{m=0}^{n} e^{-2m} = \frac{e^n \cdot e^2}{e^2-1} - \frac{e^{-n}}{e^2-1}.$$

Using the property of convolution of lattice object functions, find the object functions of the following result functions:

860. $F^*(p) = \dfrac{e^p}{(e^p-1)(e^p-e)}$.

861. $F^*(p) = \dfrac{e^{-p}}{(1-e^{-p})^2} \cdot \dfrac{e^p}{e^p-1}$.

862. $F^*(p) = \dfrac{e^p}{(e^p-1)^2(e^p-e)}$.

863. $F^*(p) = \dfrac{e^{2p}}{(e^p-e^2)(e^p-1)}$.

IX. *The result functions of differences.* The *first-order (forward) difference of the lattice function* $f(n)$ is denoted by

$\Delta f(n)$ and defined by the following expression:

$$\Delta f(n) = f(n+1) - f(n). \tag{21.17}$$

The *second-order (forward) difference* $\Delta^2 f(n)$ is defined thus:

$$\Delta^2 f(n) = \Delta f(n+1) - \Delta f(n), \tag{21.18}$$

or, taking (21.17) into account,

$$\Delta^2 f(n) = f(n+2) - 2f(n+1) + f(n). \tag{21.19}$$

In general, the *kth order difference* $\Delta^k f(n)$ is defined in the following way:

$$\Delta^k f(n) = \Delta^{k-1} f(n+1) - \Delta^{k-1}(n), \tag{21.20}$$

or

$$\Delta^k f(n) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(n+k-j), \tag{21.21}$$

where $\binom{k}{j} = \frac{k!}{j!\,(k-j)!}$ are binomial coefficients.

Example 11. Find the differences for $f(n) = 2n^2$.

Solution. By definition, the first-order difference is

$$\Delta f(n) = 2(n+1)^2 - 2n^2 = 2(2n+1),$$

and the second-order difference is

$$\Delta^2 f(n) = \Delta f(n+1) - \Delta f(n)$$
$$= 2[2(n+1)+1] - 2(2n+1) = 4.$$

Obviously, the higher-order differences are all zero (cf. the derivatives of $2t^2$).

In the following problems find the kth order differences:

864. $f(n) = 3n + 2$. **865.** $f(n) = e^{2n}$. **866.** $f(n) = n^2 - n$.

Suppose $f(n) \doteq F^*(p)$. Then

$$\Delta f(n) \doteq (e^p - 1) F^*(p) - e^p f(0),$$
$$\Delta^2 f(n) \doteq (e^p - 1)^2 F^*(p) - e^p (e^p - 1) f(0) - e^p \Delta f(0),$$

etc. In general,

$$\Delta^k f(n) \doteq (e^p - 1)^k F^*(p) - e^p \sum_{\nu=0}^{k-1} (e^p - 1)^{k-\nu-1} \Delta^\nu f(0), \tag{21.22}$$

where we have defined $\Delta^0 f(0) = f(0)$. From (21.22) we find that

$$F^*(p) = \frac{e^p}{e^p-1}\sum_{\nu=0}^{k-1}\frac{\Delta^\nu f(0)}{(e^p-1)^\nu} + \frac{1}{(e^p-1)^k}\mathcal{L}_D\{\Delta^k f(n)\}, \quad (21.23)$$

where $\mathcal{L}_D\{\Delta^k f(n)\}$ is the discrete Laplace transform (the result function) of $\Delta^k f(n)$.

If, for one, $\Delta^\nu f(0) = 0$ ($\nu = 0, 1, \ldots, k-1$) or, which is the same, $f(0) = f(1) = \ldots = f(k-1) = 0$, then formula (21.22) takes the simple form

$$\Delta^k f(n) \doteq (e^p - 1)^k F^*(p),$$

i.e. finding the kth order difference of an object function corresponds to multiplying the result function by $(e^p - 1)^k$.

Example 12. Find the result function of the lattice function $f(n) = n^2$.

Solution. We have

$$\Delta f(n) = 2n + 1,$$
$$\Delta^2 f(n) = 2n + 3 - 2n - 1 = 2,$$
$$\Delta^k f(n) = 0 \quad (k > 2).$$

Moreover, $f(0) = 0$, $\Delta f(0) = 1$, and $\Delta^2 f(0) = 2$. Putting $k = 3$ in (21.23), we find that

$$f(n) \doteq F^*(p) = \frac{e^p}{e^p-1}\sum_{\nu=0}^{2}\frac{\Delta^\nu f(0)}{(e^p-1)}$$
$$= \frac{e^p}{e^p-1}\left(0 + \frac{1}{e^p-1} + \frac{2}{(e^p-1)^2}\right) = \frac{e^p(e^p+1)}{(e^p-1)^3}.$$

Find the result functions of the following lattice functions:

867. $f(n) = n^3$. 868. $f(n) = \frac{n(n-1)}{2!}$.

869. $f(n) = (n-k)^2\,\eta(n-k)$.

X. *The result function of a sum of lattice functions.* Suppose $f(n)$ is a lattice function whose result function is $F^*(p)$.

Let us consider the sum

$$\sum_{m=0}^{n-1} f(m).$$

Then

$$\sum_{m=0}^{n-1} f(m) \doteq \frac{F^*(p)}{e^p-1},$$

i.e. finding the sum of object lattice functions corresponds to dividing the result function of $f(n)$ by $e^p - 1$. In general, summing object lattice functions k times is equivalent to dividing the result function of $f(n)$ by $(e - 1)^k$.

Example 13. Using Property X, find the sum

$$\sum_{m=0}^{n-1} me^m.$$

Solution. The result function of ne^n is $e^{p+1}/(e^p - e)^2$. Whence, by Property X_a

$$\sum_{m=0}^{n-1} me^m \doteq \frac{e^{p+1}}{(e^p-e)^2(e^p-1)}$$

$$= \frac{e}{(e-1)^2}\left[\frac{e^p}{e^p-1} - \frac{e^p}{e^p-e} + \frac{e-1}{e}\cdot\frac{e^{p+1}}{(e^p-e)^2}\right]$$

$$\doteq \frac{e}{(e-1)^2}\left[1 - e^n + \frac{(e-1)ne^n}{e}\right].$$

Consequently,

$$\sum_{m=0}^{n-1} me^m = \frac{e(1-e^n)+(e-1)ne^n}{(e-1)^2}.$$

Find the following sums:

870. $\sum_{m=0}^{n-1} m^2$. 871. $\sum_{m=0}^{n-1} m\cos m\alpha$.

872. $\sum_{m=0}^{n-1} m(m-1)$. 873. $\sum_{m=0}^{n-1} e^m \cos m\alpha \quad (n \geqslant 2)$.

The inversion formula

Let a lattice function $f(n)$ have a result function $F^*(p)$, which is a function of a complex variable $p = \sigma + i\tau$ and, in view of its periodicity, i.e. $F^*(p + 2\pi i) = F^*(p)$, is considered in its principal strip $-\pi < \operatorname{Im} p \leqslant \pi$. If we know the result function $F^*(p)$, the object function $f(n)$ can be found by the inversion formula

$$f(n) = \frac{1}{2\pi i} \int_{c-i\pi}^{c+i\pi} F^*(p)\, e^{np}\, dp, \tag{21.24}$$

where c is any real number greater than the abscissa of convergence s^*. When $F^*(p)$ is a proper rational fraction with respect to e^p, we have

$$f(n) = \sum_{\nu} \operatorname*{Res}_{p_\nu} [F^*(p)\, e^{(n-1)p}], \tag{21.25}$$

where the sum is over all the poles of $F^*(p)$ in the strip $-\pi < \operatorname{Im} p \leqslant \pi$ and on its boundary $\operatorname{Im} p = \pi$. If p_ν is a simple pole, then

$$\operatorname*{Res}_{p_\nu} [F^*(p)\, e^{p(n-1)}] = \lim_{p \to p_\nu} [F^*(p)\, (e^p - e^{p_\nu})\, e^{p(n-1)}] \tag{21.26}$$

if p_ν is an r_ν-order pole, then

$$\operatorname*{Res}_{p_\nu} [F^*(p)\, e^{p(n-1)}]$$

$$= \frac{1}{(r_\nu - 1)!} \lim_{p \to p_\nu} \frac{d^{r_\nu - 1}}{de^{p(r_\nu - 1)}} [F^*(p)\, (e^p - e^{p_\nu})^{r_\nu} e^{p(n-1)}]. \tag{21.27}$$

Example 14. Using formula (21.25), find the object function whose result function is

$$F^*(p) = \frac{e^p}{e^{2p} - 3e^p + 2}.$$

Solution. We start by finding the zeros of the denominator. We have $e^{2p} - 3e^p + 2 = 0$, whence $e_p = 1$ and $e_p = 2$. This means that $p_1 = 0$ and $p_2 = \ln 2$ are simple zeros of the denominator and, hence, they constitute all the poles of $F^*(p)$ in the principle strip and are simple. Now we have

to find the residues of $F^*(p)\,e^{p(n-1)}$ at these poles. We have

$$\operatorname*{Res}_{p=0}\left[F^*(p)\,e^{p(n-1)}\right]=\lim_{p\to 0}\left[\frac{e^p(e^p-1)}{(e^p-1)(e^p-2)}\,e^{p(n-1)}\right]$$

$$=\lim_{p\to 0}\frac{e^{pn}}{e^p-2}=-1,$$

$$\operatorname*{Res}_{p=\ln 2}\left[F^*(p)\,e^{p(n-1)}\right]=\lim_{p\to\ln 2}\frac{e^p(e^p-e^{\ln 2})\,e^{p(n-1)}}{(e^p-1)(e^p-2)}$$

$$=\lim_{p\to\ln 2}\frac{e^{pn}}{e^p-1}=\frac{e^{n\ln 2}}{e^{\ln 2}-1}=\frac{e^{\ln 2^n}}{2-1}=2^n.$$

Consequently, by (21.25) we have

$$f(n)=-1+2^n.$$

Example 15. By means of the inversion formula find the object function of

$$F^*(p)=\frac{e^p}{e^{2p}-1}.$$

Solution. The function

$$F^*(p)=\frac{e^p}{(e^p-1)(e^p+1)}$$

has two simple poles at points $p_1=0$ and $p_2=i\pi$ in the principal strip $-\pi<\operatorname{Im}p\leqslant\pi$. We find that

$$\operatorname*{Res}_{p=0}\left[F^*(p)\,e^{p(n-1)}\right]=\lim_{p\to 0}\frac{e^p(e^p-1)\,e^{p(n-1)}}{(e^p-1)(e^p+1)}=\frac{1}{2},$$

$$\operatorname*{Res}_{p=i\pi}\left[F^*(p)\,e^{p(n-1)}\right]=\lim_{p\to i\pi}\frac{e^p(e^p-e^{i\pi})\,e^{p(n-1)}}{(e^p-1)(e^p+1)}$$

$$=\lim_{p\to i\pi}\frac{e^{pn}}{e^p-1}=\frac{e^{i\pi n}}{e^{i\pi}-1}=\frac{(-1)^{n-1}}{2}.$$

According to formula (21.25),

$$f(n)=\frac{1}{2}+\frac{1}{2}(-1)^{n-1}.$$

Find the object functions for the following result functions:

874. $F^*(p)=\dfrac{e^p}{(e^p-3)^2}$. 875. $F^*(p)=\dfrac{e^p}{e^{2p}-7e^p+10}$.

876. $F^*(p) = \frac{e^p}{e^{2p}+1}$. 877. $F^*(p) = \frac{e^{2p}}{e^{4p}-1}$.

878. $F^*(p) = \frac{e^p}{e^{2p}+2ae^p+2a^2}$ $(a>0)$.

879. $F^*(p) = \frac{e^p}{(e^p-e)^2}$.

Solving difference equation via the discrete Laplace transformation

An equation of the form

$$F(n, f(n), f(n+1), \ldots, f(n+k)) = 0 \quad (21.28)$$

or

$$F(n, f(n), \Delta f(n), \ldots, \Delta^k f(n)) = 0, \quad (21.29)$$

where $f(n)$ is the unknown lattice function, is called the *kth order difference equation*. Equation (21.28) can be reduced to Eq. (21.29) via the following formula:

$$f(n+k) = f(n) + \binom{k}{1}\Delta f(n) + \binom{k}{2}\Delta^2 f(n) + \ldots \Delta^k f(n), \quad (21.30)$$

while the formula

$$\Delta^k f(n) = f(n+k) - \binom{k}{1} f(n+k-1) + \binom{k}{2} f(n+k-2) - \ldots + (-1)^k f(n) \quad (k = 0, 1, 2, \ldots), \quad (21.31)$$

where $\binom{k}{m} = \frac{k(k-1)\ldots(k-m+1)}{m!}$ are binomial coefficients, reduces Eq. (21.29) to Eq. (21.28).

If Eq. (21.29) is linear in the function $f(n)$ and the differences, it is called a *linear difference equation*. A kth order linear difference equation with constant coefficients has the form

$$b_0\Delta^k f(n) + b_1\Delta^{k-1}(n) + \ldots + b_k f(n) = \varphi(n), \quad (21.32)$$

where $\varphi(n)$ is a given lattice function, $f(n)$ the unknown lattice function, and $b_0, b_1, \ldots, b_k$ constants, with $b_0 \neq 0$ and $b_k \neq 0$. Substituting the differences $\Delta^m f(n)$ $(m = 1, 2, \ldots, k)$ from (21.31) into Eq. (21.32), we arrive

at another form of the linear difference equation:

$$a_0 f(n+k) + a_1 f(n+k-1) + \ldots + a_k f(n) = \varphi(n). \tag{21.33}$$

If $\varphi(n) \equiv 0$, Eqs. (21.32) and (21.33) are said to be *homogeneous*, while if $\varphi(n) \not\equiv 0$, these equations are said to be *nonhomogeneous*.

Since a difference equation containing $f(n)$ and $f(n+k)$ is called a difference equation of order k ($k > 0$), Eq. (21.33) with $a_0 \neq 0$ and $a_k \neq 0$ is a kth order nonhomogeneous linear difference equation. We note that the order of an equation may not coincide with the highest-order difference in this equation if it is written in form (21.32).

Example 16. The difference equation

$$\Delta^3 f(n) + 4\Delta^2 f(n) + 5\Delta f(n) + 2f(n) = 0$$

is reduced to the form

$$f(n+3) + f(n+2) = 0,$$

or

$$f(n+1) + f(n) = 0,$$

if we substitute the differences in it via (21.31). We see that it is a first-order equation.

Determine the order of each of the following equations:

880. $\Delta^4 f(n) = 4\Delta^3 f(n) + 4\Delta^2 f(n) - f(n) = 0$.

881. $\Delta^3 f(n) + 3\Delta^2 f(n) + 3\Delta f(n) + f(n) = n^3 + 1$.

882. $\Delta^3 f(n) + 2\Delta^2 f(n) + \Delta f(n) = 2^n$.

883. $\Delta^4 f(n) + 4\Delta^3 f(n) + 6\Delta^2 f(n) + 5\Delta f(n) + 2f(n) = \sin \frac{n\pi}{6}$.

For a kth order differential equation the initial conditions are given in the form of the values of the lattice function $f(n)$ and its derivatives up to the $(k-1)$st order inclusive at $n = 0$ if the equation is in form (21.32), or in the form of the values of $f(n)$ at $n = 0, 1, \ldots, k-1$ if the equation is in form (21.33).

Solution of linear difference equations with constant coefficients by means of the discrete Laplace transformation follows the same lines as that of linear differential equations with constant coefficients via the classical Laplace transfor-

mation. By applying the discrete Laplace transformation and employing the properties of linearity, translation of object function, and/or result functions of differences we arrive at a simple linear algebraic equation for the result function $F^*(p)$ of the unknown lattice function $f(n)$ (the initial conditions are incorporated into this equation). Solving this equation, we obtain the operator solution of the difference equation, and the object function of this solution is the sought solution of the difference equation that satisfies the given initial conditions.

Example 17. Find the solution of the equation

$$f(n+1) - ef(n) = 1, \quad f(0) = 0. \tag{21.34}$$

Solution. We assume that the result function of $f(n)$ is $F^*(p)$. By the property of translation of object function,

$$f(n+1) \doteq e^p F^*(p).$$

Applying the discrete Laplace transformation to both sides of Eq. (21.34), we arrive at the operator equation

$$e^p F^*(p) - eF^*(p) = \frac{e^p}{e^p - 1},$$

whence

$$F^*(p) = \frac{e^p}{(e^p - e)(e^p - 1)}.$$

The function $F^*(p)$ has two (simple) poles at $p = 0$ and $p = 1$. This means that

$$\operatorname*{Res}_{p=0} [F^*(p) e^{(n-1)p}] = \lim_{p \to 0} \frac{e^p}{e^p - e} e^{(n-1)p} = \frac{1}{1-e},$$

$$\operatorname*{Res}_{p=1} [F^*(p) e^{(n-1)p}] = \frac{e}{e-1} e^{n-1} = \frac{e^n}{e-1}.$$

Consequently, by formula (21.25),

$$f(n) = \frac{e^n - 1}{e-1}.$$

Solve the following homogeneous linear difference equations:

884. $f(n+1) - 2f(n) = 0, \ f(0) = 1.$

885. $f(n+2) + 2f(n+1) + f(n) = 0, \ f(0) = 1, \ f(1) = 0.$

886. $f(n+2) - 2f(n+1) + 2f(n) = 0, \quad f(0) = 0, \quad f(1) = 1.$

887. $f(n+3) - 3f(n+2) + 4f(n+1) - 2f(n) = 0, \quad f(0) = 0, \quad f(1) = 0, \quad f(2) = 1.$

888. $f(n+4) + f(n) = 0, \quad f(0) = 0, \quad f(1) = 1, \quad f(2) = 0, \quad f(3) = 0.$

889. $f(n+3) + 3f(n+2) + 3f(n+1) + f(n) = 0, \quad f(0) = 0, \quad f(1) = 0, \quad f(2) = 1.$

Solve the following nonhomogeneous linear difference equations:

890. $f(n+1) + 2f(n) = n, \quad f(0) = 0.$

891. $f(n+2) - 4f(n) = 4^n, \quad f(0) = f(1) = 1.$

892. $f(n+2) - f(n) = 1 - (-1)^n, \quad f(0) = 0, \quad f(1) = 1.$

893. $f(n+2) - 6f(n+1) + 9f(n) = n \cdot 3^n, \quad f(0) = 0, \quad f(1) = 0.$

894. $f(n+3) + 3f(n+2) + 3f(n+1) + f(n) = \cos n\pi, \quad f(0) = 0, \quad f(1) = 0, \quad f(2) = 0.$

895. $f(n+3) - 3f(n+2) + 3f(n+1) - f(n) = n^2, \quad f(0) = 0, \quad f(1) = 0, \quad f(2) = 0.$

Chapter 3

Stability Theory

§ 22 Stability of Solutions of Systems of Differential Equations. The Simplest Types of Equilibrium Solutions

Let us consider a system of ordinary differential equations

$$\frac{dy_i}{dt} = f_i(t, y_1, y_2, \ldots, y_n) \quad (i = 1, 2, \ldots, n), \quad (22.1)$$

where the derivatives $\partial f_i/\partial y_k$ $(i, k = 1, 2, \ldots, n)$ exist and are continuous. Suppose the $\varphi_i(t)$ $(i = 1, 2, \ldots, n)$ constitute the solution of the system that satisfies, at $t = t_0$, the following conditions:

$$\varphi_i(t_0) = \varphi_i^0 \quad (i = 1, 2, \ldots, n).$$

The solution $\varphi_i(t)$ $(i = 1, 2, \ldots, n)$ of system (22.1) is said to be *stable in the sense of Lyapunov* as $t \to +\infty$ if for every positive ε there exists a positive $\delta(\varepsilon)$ such that for each solution $y_i(t)$ $(i = 1, 2, \ldots, n)$ of the same system (22.1) whose initial values satisfy the inequalities

$$|y_i(t_0) - \varphi_i^0| < \delta(\varepsilon) \quad (i = 1, 2, \ldots, n)$$

the following inequalities are true for all $t \geqslant t_0$:

$$|y_i(t) - \varphi_i(t)| < \varepsilon \quad (i = 1, 2, \ldots, n), \quad (22.2)$$

i.e. solutions close at the initial moment remain close for all $t \geqslant t_0$. In other words, the solution $\varphi_i(t)$ $(i = 1, 2, \ldots, n)$ is stable if a solution $y_i(t)$ $(i = 1, 2, \ldots, n)$ that is close to it at the initial moment $t = t_0$ remains in an arbitrarily narrow ε-tube around the solution $\varphi_i(t)$ $(i = 1, 2, \ldots, n)$ for all $t \geqslant t_0$.

If for an arbitrarily small positive δ at least one solution $y_1(t)$ $(i = 1, 2, \ldots, n)$ does not satisfy (22.2), the solution $\varphi_i(t)$ is said to be *unstable*.

If the solution $\varphi_i(t)$ $(i = 1, 2, \ldots, n)$ is not only stable but $|y_i(t_0) - \varphi_i^0| < \delta_1$ implies

$$\lim_{t \to \infty} |y_i(t) - \varphi_i(t)| = 0 \quad (i = 1, 2, \ldots, n), \tag{22.3}$$

the solution $\varphi_i(t)$ is said to be *asymptotically stable.*

Investigation of the solution $\varphi_i(t)$ $(i = 1, 2, \ldots, n)$ of system (22.1) for stability can be reduced to the investigation for stability of the zero solution $x_i(t) \equiv 0$ of a new system of equations, which is obtained from (22.1) via a linear substitution of

$$x_i(t) = y_i(t) - \varphi_i(t) \quad (i = 1, 2, \ldots, n) \tag{22.4}$$

for the unknown functions, where the new (unknown) functions $x_i(t)$ are the deviations of the old functions $y_i(t)$ from the functions $\varphi_i(t)$, which determine the solution at the initial moment. Therefore, in what follows we will assume that it is the zero solution $x_i(t)$ that is investigated for stability or, which is the same, the equilibrium solution at the origin of coordinates of the system of equations

$$\frac{dx_i}{dt} = \psi_i(t, x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, n). \tag{22.5}$$

When applied to the *equilibrium* solution $x_i(t) \equiv 0$ $(i = 1, 2, \ldots, n)$ the stability condition is formulated thus:

The equilibrium solution $x_i(t) \equiv 0$ $(i = 1, 2, \ldots, n)$ *of system* (22.5) *is stable in the sense of Lyapunov if for every positive* ε *there exists a positive* $\delta(\varepsilon)$ *such that* $|x_i(t_0)| < \delta(\varepsilon)$ $(i = 1, 2, \ldots, n)$ *implies* $|x_i(t)| < \varepsilon$ $(i = 1, 2, \ldots, n)$ *for all* $t \geqslant t_0$.

Example 1. Prove that each solution of the equation

$$\frac{dx}{dt} = 0 \tag{22.6}$$

is stable.

Solution. Indeed, the solution $x_1(t)$ of this equation that satisfies the initial condition $x_1(t_0) = x_1^0$ is $x_1(t) \equiv x_1^0 = \text{const}$. Let us take another solution of Eq. (22.6), say $x_2(t)$, that satisfies the initial condition

$$x_2(t_0) = x_2^0. \tag{22.7}$$

We then find that $|x_2(t) - x_1(t)| = |x_2^0 - x_1^0|$ for all values of t. Therefore, for every positive ε there exists a positive δ, e.g. $\delta = \varepsilon$, such that as soon as $|x_2^0 - x_1^0|$ becomes less than δ we have

$$|x_2(t) - x_1(t)| = |x_2^0 - x_1^0| < \varepsilon \text{ for all } t \geqslant t_0.$$

Thus, every solution of Eq. (22.6) is stable. However, not one of the solutions is asymptotically stable, since

$$|x_2(t) - x_1(t)| = |x_2^0 - x_1^2| \not\to 0 \text{ as } t \to +\infty.$$

Example 2. Prove that each solution of the equation

$$\frac{dx}{dt} + x = 0 \tag{22.8}$$

is asymptotically stable.

Solution. Indeed, the general solution of Eq. (22.8) is

$$x(t) = Ce^{-t}. \tag{22.9}$$

The solutions $x_1(t)$ and $x_2(t)$ of Eq. (22.8) that satisfy the initial conditions $x_1(t_0) = x_1^0$ and $x_2(t_0) = x_2^0$ are

$$x_1(t) = x_1^0 e^{-(t-t_0)}, \quad x_2(t) = x_2^0 e^{-(t-t_0)}.$$

Whence

$$|x_2(t) - x_1(t)| = |x_2^0 - x_1^0| e^{-(t-t_0)} \to 0 \quad \text{as} \quad t \to +\infty,$$

which implies that every solution of Eq. (22.8) is asymptotically stable.

Example 3. The solution $x(t) \equiv -1$ of the equation $dx/dt = 1 - x^2(t)$ is unstable since all solutions of the equation

$$x(t) = \frac{(1+x_0)\, e^{2(t-t_0)} - (1-x_0)}{(1+x_0)e^{2(t-t_0)} + (1-x_0)}$$

tend to $+1$ as $t \to +\infty$. According to the definition the solution $x(t) \equiv 1$ of this equation is asymptotically stable.

Using the definition of Lyapunov stability, investigate the following equations and systems of equations for stability:

896. $\frac{dx}{dt} + x = 1, \qquad x(0) = 1.$

897. $\frac{dx}{dt} = -t\,(x = 1), \qquad x(0) = 1.$

898. $\frac{dx}{dt} - 2x = t, \qquad x\left(\frac{1}{2}\right) = -\frac{1}{2}.$

899. $\frac{dx}{dt} = 2xt, \qquad x(0) = 0.$

900. $\frac{dx}{dt} = \cos t, \qquad x(0) = 1.$

901. $\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -3y - 2x, \end{cases} \qquad x(0) = y(0) = 0.$

902. $\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = 2y + 3x, \end{cases} \qquad x(0) = y(0) = 0.$

The simplest types of equilibrium solutions

Let us take a system of two differential equations of the form

$$\begin{cases} \frac{dx}{dt} = P(x, y), \\ \frac{dy}{dt} = Q(x, y). \end{cases} \tag{22.10}$$

The point (x_0, y_0) in the phase plane is called an *equilibrium point* or *critical point* or *singular point* admitting a stable or unstable equilibrium solution of system (22.10) if $P(x_0, y_0) = 0$ and $Q(x_0, y_0) = 0$.

Now let us take a particular case of (22.10),

$$\begin{cases} \frac{dx}{dt} = a_{11}x + a_{12}y, \\ \frac{dy}{dt} = a_{21}x + a_{22}y, \end{cases} \tag{22.11}$$

where a_{ij} $(i, j = 1, 2)$ are constants. The point $(0, 0)$ is the critical point (equilibrium solution) of (22.11). We will study the nature of the phase trajectories of system (22.11) in the neighborhood of this point. We look for the solution of (22.11) in the form

$$x = \alpha_1 e^{kt}, \quad y = \alpha_2 e^{kt}. \tag{22.12}$$

To find k we have the following characteristic equation

$$\begin{vmatrix} a_{11}-k & a_{12} \\ a_{21} & a_{22}-k \end{vmatrix} = 0. \tag{22.13}$$

The following cases are possible.

I. The roots of the characteristic equation are real and distinct:

(a) $k_1 < 0$ and $k_2 < 0$. The critical point (equilibrium solution) is asymptotically stable (a *stable nodal point*).

(b) $k_1 > 0$ and $k_2 > 0$. The critical point (equilibrium solution) is unstable (an *unstable nodal point*).

(c) $k_1 > 0$ and $k_2 < 0$. The critical point (equilibrium solution) is unstable (a *saddle point*).

(d) $k_1 = 0$ and $k_2 > 0$. The critical point (equilibrium solution) is unstable.

(e) $k_1 = 0$ and $k_2 < 0$. The critical point (equilibrium solution) is stable but not asymptotically.

II. The roots of the characteristic equation are complex-valued: $k_1 = p + iq$ and $k_2 = p - iq$.

(a) $p < 0$ and $q \neq 0$. The critical point is asymptotically stable (a *stable focal point*).

(b) $p > 0$ and $q \neq 0$. The critical point is unstable (an *unstable focal point*).

(c) $p = 0$ and $q \neq 0$. The critical point is stable (a *vortex point*) but not asymptotically.

III. The roots are multiple: $k_1 = k_2$.

(a) $k_1 = k_2 < 0$. The critical point is asymptotically stable (a *stable nodal point*).

(b) $k_1 = k_2 > 0$. The critical point is unstable (an *unstable nodal point*).

(c) $k_1 = k_2 = 0$. The critical point is unstable.

There is the possibility of an exceptional case where all points in the plane are stable critical points.

For a system of homogeneous linear equations with constant coefficients

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j \quad (i = 1, 2, \ldots, n) \tag{22.14}$$

the characteristic equation is

$$\begin{vmatrix} a_{11}-k & a_{12} & a_{13} \dots & a_{1n} \\ a_{21} & a_{22}-k & a_{23} \dots & a_{2n} \\ \dots\dots\dots & \dots\dots\dots & & \\ a_{n1} & a_{n2} & a_{n3} \dots & a_{nn}-k \end{vmatrix} = 0. \quad (22.15)$$

(1) If the real parts of all the roots of the characteristic equation (22.15) of system (22.14) are negative, the critical point (equilibrium solution) $x_i(t) \equiv 0$ $(i = 1, 2, \dots, n)$ is asymptotically stable.

(2) If the real part of at least one root of the characteristic equation (22.15) is positive, i.e. $\operatorname{Re} k_i = p_i > 0$, the critical point (equilibrium solution) $x_i(t) \equiv 0$ $(i = 1, 2, \dots, n)$ of system (22.14) is unstable.

(3) If the characteristic equation (22.15) has simple roots with zero real parts (i.e. roots that are either zero or purely imaginary), the critical point (equilibrium solution) $x_i(t) \equiv 0$ $(i = 1, 2, \dots, n)$ of system (22.14) is stable but not asymptotically.

Example 4. Establish the nature of the critical point (0, 0) of the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x. \end{cases}$$

Solution. Here

$$a_{11} = 0, \quad a_{12} = 1, \quad a_{21} = -1, \quad a_{22} = 0.$$

The characteristic equation is

$$\begin{vmatrix} -k & 1 \\ -1 & -k \end{vmatrix} = 0, \quad \text{or} \quad k^2 + 1 = 0.$$

The roots of the characteristic equation are purely imaginary, $k_{1,2} = \pm i$. The critical point is stable (a vortex point).

Establish the nature of the critical point (0, 0) for the following systems of equations:

903. $\begin{cases} \dot{x} = x - y, \\ \dot{y} = 2x + 3y. \end{cases}$

904. $\begin{cases} \dot{x} = 4y - x, \\ \dot{y} = -9x + y. \end{cases}$

905. $\begin{cases} \dot{x} = -2x - 3y, \\ \dot{y} = x + y. \end{cases}$

906. $\begin{cases} \dot{x} = x - 2y, \\ \dot{y} = 2y - 3x. \end{cases}$

907. $\begin{cases} \dot{x} = 3x + 2y, \\ \dot{y} = x + y. \end{cases}$

908. $\begin{cases} \dot{x} = -x + 2y, \\ \dot{y} = -2x + 3y. \end{cases}$

909. $\begin{cases} \dot{x} = -2x + y, \\ \dot{y} = -x - 4y. \end{cases}$

910. $\begin{cases} \dot{x} = 2x - y + 2z, \\ \dot{y} = 5x - 3y + 3z, \\ \dot{z} = -x - 2z. \end{cases}$

911. $\begin{cases} \dot{x} = x - 3y + 4z, \\ \dot{y} = 4x - 7y + 8z, \\ \dot{z} = 6x - 7y + 7z. \end{cases}$

For the system of two linear equations with constant coefficients

$$\begin{cases} \dot{x} = a_{11}x + a_{12}y, \\ \dot{y} = a_{21}x + a_{22}y \end{cases} \tag{22.16}$$

the characteristic equation (22.3) can be reduced to

$$k^2 + a_1 k + a_2 = 0.$$

(1) If $a_1 > 0$ and $a_2 > 0$, the zero solution of system (22.16) is asymptotically stable.

(2) If $a_1 > 0$ and $a_2 = 0$ or $a_1 = 0$ and $a_2 > 0$, the zero solution is stable but not asymptotically.

(3) In all other cases the zero solution is unstable; however at $a_1 = a_2 = 0$ there is the possibility of an exceptional case where the zero solution is stable but not asymptotically.

Example 5. Find the values of parameter α for which the zero solution of the system of equations

$$\begin{cases} \dot{x} = y, \\ \dot{y} = (\alpha - 1)x - \alpha y \end{cases}$$

is stable.

Solution. The characteristic equation for this system is

$$\begin{vmatrix} -k & 1 \\ \alpha-1 & -\alpha-k \end{vmatrix}=0,$$

or $k^2+\alpha k+1-\alpha=0$. Here $a_1=\alpha$ and $a_2=1-\alpha$. The zero solution is asymptotically stable when $\alpha>0$ and $1-\alpha>0$, i.e. when $0<\alpha<1$. The zero solution is stable but not asymptotically in two cases: (a) when $\alpha>0$ and $1-\alpha=0$, i.e. when $\alpha=1$, and (b) when $\alpha=0$ and $1-\alpha>0$, i.e. when $\alpha=0$. For all other values of α the zero solution in unstable.

Find the values of parameter α for which the zero solutions of the following systems of equations are stable:

912. $\begin{cases} \dot{x}=y, \\ \dot{y}=5\alpha x-\alpha^2 y. \end{cases}$

913. $\begin{cases} \dot{x}=-x+y, \\ \dot{y}=\alpha x-\alpha^2 y. \end{cases}$

914. $\begin{cases} \dot{x}=\alpha^2 x-3y, \\ \dot{y}=\alpha x+4y. \end{cases}$

915. $\begin{cases} \dot{x}=y+\alpha x, \\ \dot{y}=-x. \end{cases}$

916. $\begin{cases} \dot{x}=\alpha x-y, \\ \dot{y}=\alpha y-z, \\ \dot{z}=\alpha z-x. \end{cases}$

Example 6. In the plane of parameters α and β find the regions where the zero solution of the system of equations

$$\begin{cases} \dot{x}=\alpha x+(\beta-2\alpha\beta-1)\,y, \\ \dot{y}=x-\beta y \end{cases}$$

is stable.

Solution. The characteristic equation of this system is

$$\begin{vmatrix} \alpha-k & \beta-2\alpha\beta-1 \\ 1 & -\beta-k \end{vmatrix}=0,$$

or

$$k^2+(\beta-\alpha)\,k+1+\alpha\beta-\beta=0.$$

Here

$$a_1=\beta-\alpha, \quad a_2=1+\alpha\beta-\beta;$$

a_1 and a_2 are continuous functions of α and β, whence a_1 and a_2 change sign where $a_1 = a_2 = 0$, i.e. on the straight line $\beta - \alpha = 0$ and the hyperbola $1 + \alpha\beta - \beta = 0$. These curves divide the plane of parameters α and β into four regions (Fig. 42) in which a_1 and a_2 retain their signs. Let

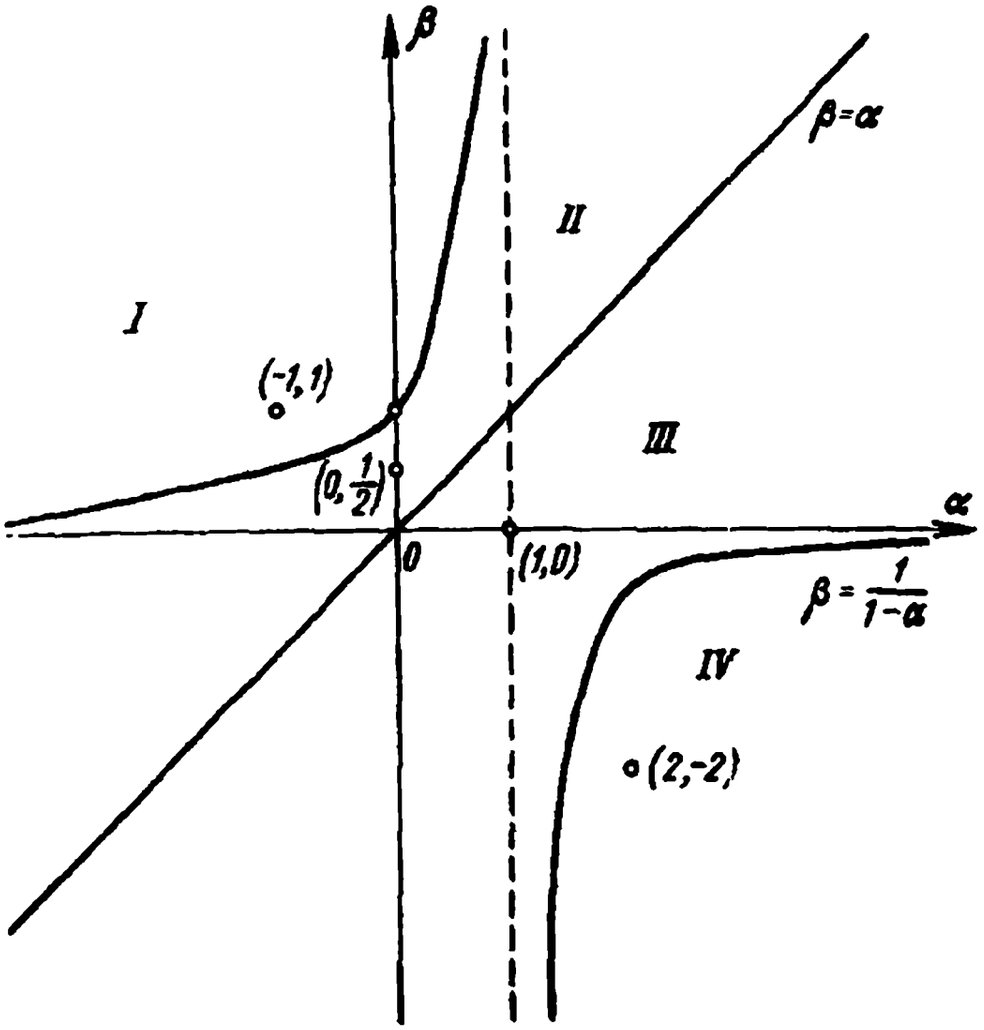


Figure 42

us select one point in each region and find the signs of a_1 and a_2 at these points.

Region I: at point $(-1, 1)$ we have $a_1 = 2 > 0$ and $a_2 = -1 < 0$. The zero solution of the system in this region is unstable.

Region II; at point $(0, 1/2)$ we have $a_1 = 1/2 > 0$ and $a_2 = 1/2 > 0$. The zero solution of the system in the region is asymptotically stable.

Region III: at point $(1, 0)$ we have $a_1 = -1 < 0$ and $a_2 = 1 > 0$. The zero solution of the system in this region is unstable.

Region IV: at point $(2, -2)$ we have $a_1 = -4 < 0$ and $a_2 = -1 < 0$. The zero solution of the system in this region is unstable.

Let us now investigate the zero solution at the boundaries of these region for stability.

(1) $\beta = 1/(1-\alpha)$ and $\alpha < 1$ (the boundary between regions I and II). At this boundary $a_1 > 0$ and $a_2 = 0$ so that the zero solution at this boundary is stable but not asymptotically.

(2) $\beta = \alpha$ (the boundary between regions II and III). At this boundary $a_1 = 0$ and $a_2 > 0$, so that the zero solution at this boundary is stable but not asymptotically.

(3) $\beta = 1/(1-\alpha)$ and $\alpha > 1$ (the boundary between regions III and IV). At this boundary $a_1 < 0$ and $a_2 = 0$, so that the zero solution at this boundary is unstable.

Thus, the zero solution is asymptotically stable in region II and is stable but not asymptotically at the boundaries of region II.

Find the regions in the plane of parameters α and β where the zero solutions of the following systems of equations are stable:

917. $\begin{cases} \dot{x} = -x + \alpha y, \\ \dot{y} = \beta x - y. \end{cases}$

918. $\begin{cases} \dot{x} = \alpha x + \beta y, \\ \dot{y} = x + \alpha y. \end{cases}$

919. $\begin{cases} \dot{x} = \alpha x + \beta y, \\ \dot{y} = -\beta x + (\alpha - 2) y. \end{cases}$

920. $\begin{cases} \dot{x} = -\alpha^2 x - \beta^2 y, \\ \dot{y} = (\alpha^2 - 1) x + (\beta^2 + 1) y. \end{cases}$

921. $\begin{cases} \dot{x} = (\alpha^2 - \beta) x + (1 + \beta) y, \\ \dot{y} = -\beta^2 x + \beta^2 y. \end{cases}$

922. $\begin{cases} \dot{x} = -\alpha^2 x - (\beta + 1) y, \\ \dot{y} = (4\alpha + \beta + 1) x - 4y. \end{cases}$

§ 23 Lyapunov's Second Method

A function $v(x_1, x_2, \ldots, x_n)$ is said to be *positive definite in an H-neighborhood* $\left(\sum_{i=1}^{n} x_i^2 \leqslant H\right)$ *of the origin of coordinates* if it is positive at all points of this neighborhood except at the origin of coordinates, where it is zero, i.e.

$$v(x_1, x_2, \ldots, x_n) > 0 \quad \text{if} \quad \sum_{i=1}^{n} x_i^2 > 0,$$
$$v(0, 0, \ldots, 0) = 0.$$

For instance, the function $v = x_1^2 + x_2^2 + x_3^2$ is positive definite in the space of variables x_1, x_2, and x_3. The function $u = x_1^2 + x_2^2$ is only of constant sign in this case but is not positive definite, since it vanishes on the x_3 axis and not only at point (0, 0, 0), but it is positive definite in the (x_1, x_2)-space.

If $v(x_1, x_2, \ldots, x_n)$ is negative if $\sum_{i=1}^{n} x_i^2 > 0$ and zero at the origin of coordinates, it is called *negative definite*.

A function $v(t, x_1, x_2, \ldots, x_n)$ is said to be *positive definite in an H-neighborhood of the origin of coordinates for all* $t \geqslant t_0$ if there exists a t-independent positive definite function $w(x_1, x_2, \ldots, x_n)$ such that $v(t, x_1, x_2, \ldots, x_n) \geqslant w(x_1, x_2, \ldots, x_n)$ for all values of the arguments and $v(t, 0, 0, \ldots, 0) = 0$.

Suppose we have a system of differential equations

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, n) \qquad (23.1)$$

and let $v(t, x_1, x_2, \ldots, x_n)$ be a continuous differentiable function of the arguments. Then the total derivative of $v(t, x_1, x_2, \ldots, x_n)$ with respect to t calculated along the integral curves of system (23.1) is

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} \frac{dx_i}{dt}$$

$$= \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i(t, x_1, x_2, \ldots, x_n). \qquad (23.2)$$

If the right-hand sides in system (23.1) do not contain t explicitly, the system is called *autonomous*.

Lyapunov's theorem on stability *If the system of differential equations* (23.1) *is such that there exists a function* $v(t, x_1, x_2, \ldots, x_n)$ *which is positive definite in an H-neighborhood of the origin of coordinates for* $t \geqslant t_0$ *and whose derivative* dv/dt *calculated along the integral curves of system* (23.1) *is nonpositive, then the zero solution of this system is stable.*

Lyapunov's theorem on asymptotic stability (for autonomous systems) *If an autonomous system of differential equations*

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, n) \qquad (23.3)$$

is such that there exists a function $v(x_1, x_2, \ldots, x_n)$ *which is positive definite in an H-neighborhood of the origin of coordinates and whose derivative* dv/dt *calculated along the integral curves of system* (23.3) *is negative definite, the zero solution* $x_i \equiv 0$ $(i = 1, 2, \ldots, n)$ *is asymptotically stable.*

The functions $v(t, x_1, x_2, \ldots, x_n)$ and $v(x_1, x_2, \ldots, x_n)$ of these theorems are known as *Lyapunov functions.*

If in a region of the H-neighborhood $\sum_{i=1}^{n} x_i^2 \leqslant H$ of the origin of coordinates in the space of variables $x_1, x_2, \ldots, x_n$ the function v assumes positive values and the region is bounded by the surface $v = 0$, the region is called a $v > 0$ *region.*

Suppose that a function v possesses the following properties:

(1) there exists a $v > 0$ region for arbitrarily large t's and in arbitrarily small neighborhoods of the origin of coordinates:

(2) in the $v > 0$ region the function v is bounded;

(3) in the $v > 0$ region the derivative dv/dt calculated along the integral curves of system (23.1) is positive definite.

Then we have

Chetaev's theorem on instability *If for the system of differential equations* (23.1) *we can find a function that satisfies conditions* (1), (2), *and* (3), *then the zero solution of this system is unstable.*

Remark. If all the f_i in system (23.1) do not depend on t explicitly, we must look for a Lyapunov function that does not depend on t explicitly either.

Example 1. Investigate the zero solution of the system

$$\begin{cases} \frac{dx}{dt} = -(x-2y)(1-x^2-3y^2), \\ \frac{dy}{dt} = -(y+x)(1-x^2-3y^2) \end{cases}$$

for stability.

Solution. We take $v = x^2 + 2y^2$. This function (a) is positive definite and (b) its derivative dv/dt taken along the integral curves of this system is

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}$$

$$= 2x(2y-x)(1-x^2-3y^2) - 4y(x+y)(1-x^2-3y^2)$$

$$= -2(1-x^2-3y^2)(x^2+2y^2) \leqslant 0$$

for small x's and y's. We see that the hypotheses of Lyapunov's theorem on stability are fulfilled. This means that the zero solution $x \equiv 0$, $y \equiv 0$ is stable.

Example 2. Investigate the zero solution of the system

$$\begin{cases} \frac{dx}{dt} = -5y - 2x^3, \\ \frac{dy}{dt} = 5x - 3y^3 \end{cases}$$

for stability.

Solution. The function $v = x^2 + y^2$ satisfies the hypotheses of Lyapunov's theorem on asymptotic stability:

(1) $v(x, y) \geqslant 0$, $v(0, 0) = 0$;

(2) $\frac{dv}{dt} = 2x(-5y - 2x^3) + 2y(5x - 3y^3) =$

$$= -(4x^4 + 6y^4) \leqslant 0,$$

i.e. $dv/dt < 0$, and $dv/dt = 0$ only when $x = 0$ and $y = 0$, which means that v is a negative definite function. Consequently, the solution $x \equiv 0$, $y \equiv 0$ is asymptotically stable.

Example 3. Investigate the zero solution of the autonomous system

$$\begin{cases} \dfrac{dx}{dt} = x^3 + y, \\ \dfrac{dy}{dt} = y^3 + x \end{cases}$$

for stability.

Solution. We take the Lyapunov function $v(x, y)$ in the form

$$v = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3.$$

Here the $v > 0$ region is $x > 0$, $y > 0$. In this region we have

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt} = (x^2 + y)^2 + (x + y^2)^2 > 0.$$

According to Chetaev's theorem on instability, the solution $x \equiv 0$, $y \equiv 0$ is unstable.

We will now give an example of building a Lyapunov function by means of the so-called *method of separation of variables.*

Example 4. Given a system of equations

$$\begin{cases} \dot{x} = ax^3 + by, \\ \dot{y} = -cx + dy^3. \end{cases} \tag{23.4}$$

We look for the Lyapunov function of system (23.4) in the form

$$v(x, y) = F_1(x) + F_2(y),$$

where $F_1(x)$ and $F_2(x)$ are differentiable functions that have yet to be found.

Solution. Bearing in mind system (23.4), we have

$$\dot{v} = F_1'(x)\dot{x} + F_2'(y)\dot{y} = F_1'(x)(ax^3 + by) - F_2'(y)(cx - dy^3)$$

We will require that $\dot{v}$ has the same form as $v(x, y)$, i.e. is the sum of two functions each of which depends only on one variable. For this to be the case we must require that

$$F_1'(x)\,by - F_2'(y)\,cx \equiv 0.$$

Separating the variables, we have

$$\frac{cx}{F_1'(x)} = \frac{by}{F_2'(y)},$$

which means that each of the quotients must be a constant, say 1/2. Then

$$\frac{cx}{F_1'(x)} = \frac{1}{2}, \quad \frac{by}{F_2'(y)} = \frac{1}{2},$$

whence

$$F_1(x) = cx^2, \quad F_2(y) = by^2,$$

which yields

$$v(x, y) = cx^2 + by^2.$$

Investigate for stability the zero solutions of the following systems of equations:

923. $\begin{cases} \dot{x} = -x + y - 3xy^2 - \frac{1}{4}x^3, \\ \dot{y} = -\frac{1}{3}x - \frac{1}{2}y - 2y^3. \end{cases}$

924. $\begin{cases} \dot{x} = y - 3x^3, \\ \dot{y} = -x - 7y^3. \end{cases}$

925. $\begin{cases} \dot{x} = -2x - y + 2xy^2 - 3x^3, \\ \dot{y} = \frac{1}{3}x - y - x^2y - 7y^3. \end{cases}$

926. $\begin{cases} \dot{x} = -xy^4, \\ \dot{y} = yx^4. \end{cases}$

927. $\begin{cases} \dot{x} = -5x - 9y + 3xy^2 - x^3, \\ \dot{y} = 3x - 4y - 2x^2y - \frac{1}{2}y^3. \end{cases}$

928. $\begin{cases} \dot{x} = -x - 2xy^2 - xy^6, \\ \dot{y} = -\frac{1}{2}y - x^2y - x^4y^3. \end{cases}$

929. $\begin{cases} \dot{x} = -3x + xy^4 - x^3y^6, \\ \dot{y} = -\frac{1}{2}x^2y - \frac{1}{4}y^6. \end{cases}$

930. $\begin{cases} \dot{x} = x - xy^4, \\ \dot{y} = y - x^2y^3. \end{cases}$

931. $\begin{cases} \dot{x} = y^3 + x^5, \\ \dot{y} = x^3 + y^5. \end{cases}$

932. $\begin{cases} \dot{x} = -2y - x^3, \\ \dot{y} = 3x - 4y^3. \end{cases}$

933. $\begin{cases} \dot{x} = -\frac{1}{2}x - \frac{1}{2}xy^2, \\ \dot{y} = -\frac{3}{4}y + 3xz^3, \\ \dot{z} = -\frac{2}{3}z - 2xyz^2 \end{cases}$

$(v = x^2 + 2y^2 + 3z^2)$.

934. $\begin{cases} \dot{x} = \frac{1}{3}y - x - \frac{7}{2}x^3, \\ \dot{y} = -x - \frac{1}{2}y - \frac{5}{2}y^3. \end{cases}$

§ 24 Stability in the First Approximation

Let us consider a system of differential equations

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, n), \tag{24.1}$$

where the f_i are functions differentiable in the neighborhood of the origin of coordinates and such that $f_i(t, 0, 0, \ldots 0) \equiv \equiv 0$. We will investigate for stability the critical point (equilibrium solution) $x_i \equiv 0$ $(i = 1, 2, \ldots, n)$ of system (24.1). We write system (24.1) in the neighborhood of the origin of coordinates in the form

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)\, x_j + R_i(t, x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, n), \tag{24.2}$$

where the R_i are of an order greater than the first with respect to $\sqrt{\sum_{i=1}^{n} x_i^2}$, i.e. we must expand the right-hand sides of (24.1) in Taylor series in powers of x in the neighborhood of the origin of coordinates. Instead of investigating the critical point of system (24.1) for stability we will investigate the critical point of the linear system

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)\, x_j \quad (i = 1, 2, \ldots, n), \tag{24.3}$$

called the *first-approximation system of equations*, or the *linearized system of equations*, for system (24.1).

But does the stability or instability of the critical point of (24.3) imply the stability or instability of the critical point of (24.1)? Generally speaking, there is no rigid interrelationship between (24.1) and (24.3).

Example 1. Take the equation

$$\frac{dx}{dt} = x^2. \tag{24.4}$$

Here $f(t, x) \equiv x^2$. The linearized equation for Eq. (24.4) is

$$\frac{dx}{dt} = 0. \tag{24.5}$$

The solution $x(t) \equiv 0$ of Eq. (24.5) is stable (see Example 1 in § 22). But this solution is not stable for Eq. (24.4). Indeed, every real solution of Eq. (24.4) has the form

$$x = \frac{x_0}{1 - tx_0} \qquad x|_{t=0} = x_0,$$

and becomes infinite at $t = 1/x_0$, i.e. the solution cannot be continued indefinitely.

Example 2. Take the nonlinear equation

$$\frac{dx}{dt} = x - x^3 e^t. \tag{24.6}$$

The linearized equation is

$$\frac{dx}{dt} = x. \tag{24.7}$$

The solution $x(t) \equiv 0$ of Eq. (24.7) is unstable, since each solution of this equation has the form

$$x(t) = Ce^t$$

and, obviously, $|x(t)| \to +\infty$ as $t \to +\infty$. On the other hand, the solution $x(t) \equiv 0$ of Eq. (24.6) is asymptotically stable. Indeed, the general solution of this equation can be written in the form

$$x(t) = Ce^t \left[1 + \frac{2}{3} C^2 (e^{3t} - 1)\right]^{-1/2}$$

and, obviously, tends to zero as $t \to +\infty$.

In certain cases, however, the stability (or instability) of the solution of the linearized system of equations implies the stability (or instability) of the solution of system (24.1). For the sake of simplicity we will consider only the case where the coefficients $a_{ij}(t)$ in (24.3) are all constant. We will then say that the system (24.2) is *autonomous in the first approximation.*

Theorem 1 *If the system of equations* (24.2) *is autonomous in the first approximation, all the* R_i *are bounded in* t *and*

can be expanded in power series in $x_1, x_2, \ldots, x_n$ *in a neighborhood* $\sum_{i=1}^{n} x_i^2 \leqslant H$, *with the expansions starting with terms whose order is no lower than the second, and all the roots of the characteristic equation*

$$\begin{vmatrix} a_{11}-k & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-k & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-k \end{vmatrix} = 0 \qquad (24.8)$$

have negative real parts, then the zero solution $x_i \equiv 0$ $(i = 1, 2, \ldots, n)$ *of system* (24.2) *is asymptotically stable, i.e. in this case it is possible to investigate for stability in the first approximation.*

Theorem 2 *If the system of equations* (24.2) *is autonomous in the first approximation, all the* R_i *satisfy the hypotheses of Theorem* 1, *and at least one of the roots of the characteristic equation* (24.8) *has a positive real part, then the critical point (equilibrium solution)* $x_i \equiv 0$ $(i = 1, 2, \ldots, n)$ *of system* (24.2) *is unstable, i.e. it is possible to investigate for stability in the first approximation.*

Remark. If the real parts of all the roots of the characteristic equation are nonpositive, with the real part of at least one root being zero, then it is not generally possible to investigate for stability in the first approximation (in this case the nonlinear terms R_i come into play).

Example 3. Investigate for stability the critical point $x = 0$, $y = 0$ of the system

$$\begin{cases} \dot{x} = -x + y + 2x^4 - y^6, \\ \dot{y} = x - 3y + 11y^4. \end{cases} \qquad (24.9)$$

Solution. The nonlinear terms satisfy the hypotheses of Theorems 1 and 2. Let us investigate for stability the critical point of the first-order approximation system

$$\begin{cases} \dot{x} = -x + y, \\ \dot{y} = x - 3y. \end{cases} \qquad (24.10)$$

The characteristic equation

$$\begin{vmatrix} -1-k & 1 \\ 1 & -3-k \end{vmatrix} = 0$$

has only negative roots, $k_{1,2} = -2 \pm \sqrt{2}$. Therefore, by Theorem 1, the critical point $x = 0$, $y = 0$ of systems (24.9) and (24.10) is asymptotically stable.

Example 4. Let us take the equation that describes the oscillations of a pendulum.

$$\ddot{x} + a\dot{x} + b \sin x = 0, \tag{24.11}$$

where x is the angle of deflection from the vertical line. Equation (24.11) is equivalent to the following system:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -ay - b \sin x. \end{cases} \tag{24.12}$$

The critical points (equilibrium solutions) for (24.12) are

$$x = k\pi \ (k \text{ an integer}), \quad y = 0. \tag{24.13}$$

Let us investigate for stability the critical point $x = 0$, $y = 0$, which we can obtain from (24.13) by putting $k = 0$. We use the expansion

$$\sin x = x - \frac{x^3}{3!} + \dots$$

and write the first-order approximation system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -bx - ay, \end{cases} \tag{24.14}$$

whose characteristic equation is

$$\lambda^2 + a\lambda + b = 0. \tag{24.15}$$

If $a > 0$ and $b > 0$, the roots of Eq. (24.15) have negative real parts and, consequently, the critical point $x = 0$, $y = 0$ is stable in the first approximation.

Now let us investigate for stability the point $(\pi, 0)$, which corresponds to $k = 1$. We use the expansion

$$\sin x = -(x - \pi) + \frac{(x-\pi)^3}{3!} - \dots$$

and shift the origin of coordinates to the point $x = \pi$, $y = 0$. We arrive at the following linearized system:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = bx - ay, \end{cases} \tag{24.16}$$

whose characteristic equation is

$$\lambda^2 + a\lambda - b = 0. \tag{24.17}$$

If $a > 0$ and $b > 0$, the roots of this equation are real and have different signs. Consequently, for system (24.16) the critical point $(\pi, 0)$ is unstable.

Example 5. Investigate for stability the critical point $x = 0$, $y = 0$ of the system

$$\begin{cases} \dot{x} = y - xf(x, y), \\ \dot{y} = -x - yf(x, y). \end{cases} \tag{24.18}$$

where the function $f(x, y)$ can be expanded in a convergent power series, and $f(0, 0) = 0$.

Solution. The linearized system is

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x. \end{cases} \tag{24.19}$$

The critical point of (24.19) is $x = 0$, $y = 0$. The characteristic equation of (24.19) is

$$\begin{vmatrix} -k & 1 \\ -1 & -k \end{vmatrix} = 0, \quad \text{or} \quad k^2 + 1 = 0, \tag{24.20}$$

and has only purely imaginary roots, $k_{1,2} = \pm i$. The critical point (0, 0) of the first-order approximation system is stable (a vortex point). Since the real parts of the roots of the characteristic equation (24.20) are zero, according to the remark after Theorem 2 the question of whether the critical point (0, 0) is stable requires additional investigation. Let us apply Lyapunov's second method to see whether the critical point (0, 0) of system (24.18) is stable. Assuming that $v(x, y) = (x^2 + y^2)/2$, we find that

$$\frac{dv}{dt} = -(x^2 + y^2) f(x, y).$$

The result is as follows: If $f(x, y) \geqslant 0$ in a small neighborhood of the origin of coordinates, the critical point (0, 0) is stable. If $f(x, y)$ is a positive definite function in a small neighborhood of the origin of coordinates, the critical point (0, 0) is asymptotically stable. Finally, if $f(x, y) < 0$ in a small neighborhood of the origin of coordinates, the critical point (0, 0) is unstable. This example illustrates the simple fact that in some cases one cannot judge the stability of a critical point solely by the first-order approximation.

Investigate for stability in the first approximation the critical point $x = 0$, $y = 0$ of the following system of equations:

935. $\begin{cases} \dot{x} = -x + 2y - 3x^2, \\ \dot{y} = 3x - 2y + 2x^2 + y^4. \end{cases}$

936. $\begin{cases} \dot{x} = -\sin x + 3y + x^5, \\ \dot{y} = \frac{1}{4}x - 2y - \frac{1}{6}y^3. \end{cases}$

937. $\begin{cases} \dot{x} = 2e^x + 5y - 2 + x^4, \\ \dot{y} = x + 6\cos y - 6 - y^2. \end{cases}$

938. $\begin{cases} \dot{x} = -3x + 4y + \sin^3 x - y^2, \\ \dot{y} = -2x + \sin y + e^y x^2. \end{cases}$

939. $\begin{cases} \dot{x} = x - 2\sin y - y^3 \sin x, \\ \dot{y} = 2y - 3x - x^3. \end{cases}$

940. $\begin{cases} \dot{x} = x - y + x^2 + y^2, \\ \dot{y} = x + y - y^2. \end{cases}$

941. $\begin{cases} \dot{x} = 2x + 8\sin y, \\ \dot{y} = 2 - e^x - 3y - \cos y. \end{cases}$

942. $\begin{cases} \dot{x} = -4x + \frac{7}{2}\sin y - 3x^2, \\ \dot{y} = -2x + x^2 + y + y^3. \end{cases}$

943. $\begin{cases} \dot{x} = -4y - x^3, \\ \dot{y} = 3x - y^3. \end{cases}$

944. $\begin{cases} \dot{x} = 10\sin x - 29y + 3y^3, \\ \dot{y} = 5x - 14\sin y + y^2. \end{cases}$

945. $\begin{cases} \dot{x} = y - xy^2, \\ \dot{y} = -x^3. \end{cases}$

946. Investigate for stability the critical points of the equation of oscillations of a pendulum with a torque L applied to it:

$$\ddot{x} + a\dot{x} + b\sin x = L, \quad \text{where } |L| < b.$$

§ 25 Asymptotic Stability in the Large. Lagrange Stability

Suppose we have a system of differential equations

$$\frac{dx_i}{dt} = f_i(t, x_1, \ldots, x_n), \quad f_i(t, 0, \ldots, 0) = 0 \quad (i = 1, 2, \ldots, n) \tag{25.1}$$

defined in the half-space

$$\Omega : \{a < t < +\infty, \ \sum_{i=1}^{n} x_i^2 < +\infty\}.$$

We say that the zero solution $x_i = 0$ $(i = 1, 2, \ldots, n)$ of system (25.1) is *asymptotically stable in the large* if

(1) it is asymptotically stable in the sense of Lyapunov;

(2) any other solution $x_i(t)$ $(i = 1, 2, \ldots, n)$ of system (25.1) is such that

$$\lim_{t \to \infty} x_i(t) = 0 \qquad (i = 1, 2, \ldots, n).$$

The asymptotic stability in the large for a nonzero solution of system (25.1) is defined similarly.

We will restrict our discussion to autonomous systems, i.e. systems of equations whose right-hand sides are time-independent:

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n), \quad f_i(0, \ldots, 0) = 0 \quad (i = 1, \ldots, n). \tag{25.2}$$

We will say that a Lyapunov function $v(x_1, \ldots, x_n)$ is *infinitely large* if for each positive M there exists a positive R such that outside the sphere $\sum_{i=1}^{n} x_i^2 = R^2$ we have $v > M$.

The theorem on asymptotic stability in the large *If there exists an infinitely large positive definite function* $v(x_1, \ldots, x_n)$ *such that* $dv/dt < 0$ *outside* E *and* $dv/dt \geqslant 0$ *on* E, *where the set* E *does not contain entire trajectories (except the zero equilibrium position), then the zero solution of system* (25.2) *is asymptotically stable in the large.*

Example 1. Consider the equation

$$\ddot{x} + x^2\dot{x} + x^3 = 0. \tag{25.3}$$

The system of equations equivalent to Eq. (25.3) is

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x^3 - x^2y. \end{cases} \tag{25.4}$$

Take the Lyapunov function in the form

$$v(x, y) = \frac{1}{2}y^2 + \frac{1}{4}x^4.$$

We have

$$\frac{dv}{dt} = y\dot{y} + x^3\dot{x} = -x^3y - x^2y^2 + x^3y = -x^2y^2.$$

Obviously, $v(x, y) \to \infty$ as $x^2 + y^2 \to \infty$. Moreover, $\dot{v}(x, y)$ vanishes only on the coordinate axes (set E). Not one of the solutions except the zero solution remains on these axes for all $t \geqslant 0$. Indeed, the gradient

$$\frac{dy}{dx} = \frac{-x^3 - x^2y}{y}$$

has finite values at all points of the y axis except the origin of coordinates, which means that a section of a trajectory does not lie on this axis. On the other hand, as the x axis is approached the gradient dy/dx tends to ∞, which means that a section of a trajectory does not lie on this axis. Consequently, the set E does not contain entire trajectories (except the origin of coordinates). In view of the above theorem, point (0, 0) is asymptotically stable in the large.

947. Show that if the zero solution of a linear autonomous system is asymptotically stable in the sense of Lyapunov, it is asymptotically stable in the large.

Investigate for asymptotic stability in the large the zero solutions of the following equations (see *Barbashin*):

948. $\ddot{x} + \dot{x}^3 + (\dot{x}^2 + 1)\, x = 0.$

949. $\ddot{x} + \dot{x} + (\dot{x}^2 + \dot{x} + 2)\,(2\dot{x} + \dot{x}^5) = 0.$

950. $\ddot{x} + x^2 e^{-x}\dot{x} + x^3 + 2x = 0.$

It may prove to be that system (25.2) does not have solutions that are stable everywhere but, nevertheless, there may be regions of asymptotic stability. A *region of asymptotic stability* for solutions of system (25.2) is one that contains the origin of coordinates and possesses the property that all trajectories starting in this region tend to the origin of coordinates as $t \to \infty$.

In linear systems there may only be total stability, while for nonlinear systems this may not be the case.

Theorem *Suppose* $v(x_1, \ldots, x_n)$ *is a function that has continuous first-order partial derivatives with respect to all* x_i. *Let us denote the set of all points where* $v(x_1, \ldots, x_n) < l$ *by* Ω_l. *If the set* Ω_l *is bounded and in it*

(1) $v(x_1, \ldots, x_n) > 0$ *for* $x_i \neq 0$, $(i = 1, 2, \ldots, n)$,

(2) $\dot{v}(x_1, \ldots, x_n) < 0$ *for* $x_i \neq 0$

then the origin of coordinates constitutes a solution of system (25.2) *that is asymptotically stable, and* Ω_l *is the region of asymptotic stability.*

Example 2. Find the region of asymptotic stability for solutions of Van der Pol's differential equation

$$\ddot{x} + \varepsilon (x^2 - 1)\, \dot{x} + x = 0 \quad (\varepsilon < 0).$$

Solution. Van der Pol's equation is equivalent to the following system of equations:

$$\begin{cases} \dot{x} = y - \varepsilon\left(\frac{x^3}{3} - x\right), \\ \dot{y} = -x. \end{cases}$$

The only critical point of this system is the origin of coordinates. We take the Lyapunov function in the form $v(x, y) = (x^2 + y^2)/2$. Then $\dot{v}(x, y) = y\dot{y} + x\dot{x} = -\varepsilon x^2\left(\frac{x^2}{3} - 1\right)$. Obviously, $\dot{v} \leqslant 0$ for $x^2 \leqslant 3$ ($\varepsilon < 0$). Thus, in the circle $x^2 + y^2 < 3$ we have $v > 0$ and $\dot{v} < 0$ for $x^2 + y^2 \neq 0$, i.e. this circle lies in the region of asymptotic stability.

Lagrange stability

Suppose we have a system of differential equations

$$\frac{dx_i}{dt} = f_i(t, x_1, \ldots, x_n) \quad (i = 1, 2, \ldots, n), \qquad (25.5)$$

where $f_i(t, x_1, \ldots, x_n)$ satisfies the hypotheses of the theorem on the existence and uniqueness of the solution of system (25.5) for all $t \in [t_0, +\infty)$ and all values of $x_1, \ldots, x_n$.

Definition. System (25.5) is said to be *stable in the sense of Lagrange*, or simply *Lagrange stable*, if all solutions of this system are defined and bounded in $[t_0, +\infty)$.

951. Show that all solutions of

$$\ddot{x}(t) + \left(a^2 + \frac{1}{1+t^2}\right) x(t) = 0$$

are bounded in $[1, +\infty)$.

952. Show that all solutions of

$$\ddot{x}(t) + \left(1 + e^{-t^2} - \frac{1}{t+2}\right) x(t) = 0$$

are bounded in $[1, +\infty)$.

953. Show that the fact that all solutions of the "limiting" equation $\ddot{x}(t) + x(t) = 0$ are bounded does not mean that

all solutions of the equations

$$\text{(a) } \ddot{x}(t) - \frac{2}{t}\dot{x}(t) + x(t) = 0; \qquad \text{(b) } \ddot{x}(t) + \frac{2}{t}\dot{x}(t) + x(t) = 0$$

are bounded, too.

§ 26 The Routh-Hurwitz Criterion

Great practical use can be made of the necessary and sufficient conditions that all the roots of the algebraic equation with real coefficients $a_0, a_1, a_2, \ldots, a_n$,

$$f(\lambda) \equiv a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0, \quad (26.1)$$

have negative real parts. Without loss of generality we can assume that a_0 is positive.

The fact that all the coefficients are positive is a necessary but not sufficient condition that all the roots of Eq. (26.1) lie to the left of the imaginary axis (this is both necessary and sufficient for first- and second-degree equations). The necessary and sufficient conditions that the real parts of the roots of Eq. (26.1) are negative were given independently by Routh and Hurwitz.

The Routh-Hurwitz criterion *All the roots of Eq.* (26.1) *have negative real parts if and only if all the principal diagonal minors of Hurwitz's matrix*

$$\begin{pmatrix} a_1 & a_0 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & a_0 & \ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & a_n \end{pmatrix} \quad (26.2)$$

are positive.

A polynomial $f(\lambda)$ of a degree $n \geqslant 1$ is said to be *stable* if all its roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ have negative real parts, $\operatorname{Re}\lambda_j < 0$ $(j = 1, 2, \ldots, n)$, i.e. all the roots of a stable polynomial lie in the left half-plane.

A Hurwitz matrix is constructed in the following way. We place the coefficients of Eq. (26.1) a_1 to a_n on the principal diagonal. The columns are then filled with coefficients that have only odd indices or only even, with index 0 being considered even. All the missing elements, i.e. coefficients

whose indices are greater than n or less than 0, are assumed to be zeros. This means that the principal diagonal minors of Hurwitz's matrix are

$$\Delta_1 = a_1, \quad \Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \ldots,$$

$$\Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & \ldots & 0 \\ a_5 & a_4 & a_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a_n \end{vmatrix}. \tag{26.3}$$

Therefore, the Routh-Hurwitz criterion can be written as

$$\Delta_1 > 0, \quad \Delta_2 > 0, \ldots, \quad \Delta_n > 0. \tag{26.4}$$

Note that since $\Delta_n = \Delta_{n-1} a_n$, the last condition $\Delta_n > 0$, can be substituted for by $a_n > 0$.

Example 1. Investigate for stability the zero solution of the equation

$$y^{(V)} + y^{(IV)} + 7y''' + 4y'' + 10y' + 3y = 0.$$

Solution. The characteristic equation is

$$f(\lambda) = \lambda^5 + \lambda^4 + 7\lambda^3 + 4\lambda^2 + 10\lambda + 3 = 0.$$

Here $a_0 = 1$, $a_1 = 1$, $a_2 = 7$, $a_3 = 4$, $a_4 = 10$, and $a_5 = 3$. The principal diagonal minors of Hurwitz's matrix are

$$\Delta_5 = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 4 & 7 & 1 & 1 & 0 \\ 3 & 10 & 4 & 7 & 1 \\ 0 & 0 & 3 & 10 & 4 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} = 3\Delta_4 = 3 \times 8 = 24 > 0,$$

$$\Delta_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 4 & 7 & 1 & 1 \\ 3 & 10 & 4 & 7 \\ 0 & 0 & 3 & 10 \end{vmatrix} = 10\Delta_3 - 3 \begin{vmatrix} 1 & 1 & 0 \\ 4 & 7 & 1 \\ 3 & 10 & 7 \end{vmatrix}$$

$$= 50 - 3(49 + 3 - 10 - 28) = 8 > 0$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 0 \\ 4 & 7 & 1 \\ 3 & 10 & 4 \end{vmatrix} = 5 > 0, \quad \Delta_2 = \begin{vmatrix} 1 & 1 \\ 4 & 7 \end{vmatrix} = 3 > 0, \quad \Delta_1 = 1 > 0.$$

Thus $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 > 0$, $\Delta_4 > 0$, and $\Delta_5 > 0$. Therefore, the zero solution $y \equiv 0$ of the differential equation is asymptotically stable.

The calculations can be done in the following manner. First calculate the minor Δ_n. Then calculate Δ_1, Δ_2, Δ_3, etc. If there is a negative minor in this sequence, the solution is unstable and all other calculations are unnecessary.

If the coefficients of Eq. (26.1) are given in number form, criterion (26.4) can easily be verified. But if the elements of (26.2) contain parameters that can vary, the calculation of the minors for large k's becomes intricate.

It can be shown that if the criterion (26.4) is met, then all the coefficients in the polynomial in (26.1) are positive, or

$$a_0 > 0, \ a_1 > 0, \ \ldots, \ a_n > 0. \tag{26.5}$$

As noted earlier, this is necessary but not sufficient for the roots of $f(\lambda)$ to lie in the left half-plane $\operatorname{Re} \lambda < 0$. But if (26.5) is verified, the inequalities in (26.4) cease to be independent. For instance, at $n = 5$ the Routh-Hurwitz criterion leads to two inequalities, $\Delta_2 > 0$ and $\Delta_4 > 0$. This enabled Liénard and Chipart to establish another criterion for stability, in which the number of inequalities involving determinants is less by approximately a factor of two in comparison with criterion (26.4).

The Liénard-Chipart test *All the roots of the polynomial*

$$f(\lambda) \equiv a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n \tag{26.1'}$$

have negative real parts if and only if (1) *all the coefficients in* $f(\lambda)$ *are positive,*

$$a_0 > 0, \ a_1 > 0, \ a_2 > 0, \ \ldots, \ a_n > 0,$$

and (2)

$$\Delta_{n-1} > 0, \ \Delta_{n-3} > 0, \ \ldots \tag{26.6}$$

(just as in (26.4), Δ_k is the k-rowed principal diagonal minor of Hurwitz's matrix).

Example 2. Let us take the same equation as in Example 1:

$$y^{(V)} + y^{(IV)} + 7y''' + 4y'' + 10y' + 3y = 0.$$

Here

$$a_0 = a_1 = 1 > 0,\ a_2 = 7 > 0,\ a_3 = 4 > 0,$$
$$a_4 = 10 > 0,\ a_5 = 3 > 0,$$

i.e. the first part of the Liénard-Chipart test is satisfied. Moreover,

$$\Delta_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 4 & 7 & 1 & 1 \\ 3 & 10 & 4 & 7 \\ 0 & 0 & 3 & 10 \end{vmatrix} = 8 > 0, \quad \Delta_2 = \begin{vmatrix} 1 & 1 \\ 4 & 7 \end{vmatrix} = 3 > 0,$$

i.e. the second part is satisfied too. Thus, the zero solution of the differential equation is asymptotically stable.

Investigate for stability the zero solutions of the following equations:

954. $y^{(IV)} + 7y''' + 12y'' + 23y' + 10y = 0.$

955. $y^{(IV)} + 2y''' + 4y'' + 2y' + 5y = 0.$

956. $y^{(IV)} + y''' + 3y'' + 2y' + y = 0.$

957. $y^{(V)} + 2y^{(IV)} + 3y''' + 2y'' + y' + y = 0.$

Find the values of α that will ensure that the zero solutions of the equations below are stable.

958. $y''' + \alpha y'' + 2y' + y = 0.$

959. $y^{(IV)} + 2y''' + y'' + \alpha y' + 3y = 0.$

960. $y^{(IV)} + 3y''' + \alpha y'' + 2y' + y = 0.$

Find the values of α and β that will ensure that the zero solutions of the equations below are stable.

961. $y''' + \alpha y'' + 2y' + \beta y = 0.$

962. $y''' + \alpha y'' + \beta y' + 3y = 0.$

963. $y^{(IV)} + \alpha y''' + 2y'' + \beta y' + y = 0.$

964. Find the Routh-Hurwitz criterion for the case of the reciprocal equation

$$\lambda^4 + p\lambda^3 + q\lambda^2 + p\lambda + 1 = 0$$

(p and q are real).

§ 27 Geometric Criterion of Stability (Mikhailov's Criterion)

Let us take the nth order differential equation with constant real coefficients

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0. \qquad (27.1)$$

To establish the stability of a solution of Eq. (27.1) we must find the location of the roots of the characteristic equation

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n = 0 \qquad (27.2)$$

in the complex plane. This is done by applying *Mikhailov's criterion*, which we will now discuss.

The characteristic polynomial is

$$f(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n. \qquad (27.3)$$

Substitution of $i\omega$ for λ yields

$$f(i\omega) = u(\omega) + iv(\omega), \qquad (27.4)$$

where

$$\left.\begin{aligned} u(\omega) &= a_n - a_{n-2}\omega^2 + a_{n-4}\omega^4 - \ldots, \\ v(\omega) &= a_{n-1}\omega - a_{n-3}\omega^3 + a_{n-5}\omega^5 - \ldots. \end{aligned}\right\} \qquad (27.5)$$

According to (27.4) and (27.5), we can depict $f(i\omega)$ by a vector in the complex uOv plane. Varying the parameter ω from $-\infty$ to $+\infty$ makes the terminal point of the vector trace out a curve, called the *hodograph of vector* $f(i\omega)$ each point of which corresponds to a definite value of ω. This curve is known as *Mikhailov's curve for the polynomial* $f(\lambda)$ (Fig. 43).

As ω changes from $-\infty$ to $+\infty$ the vector $f(i\omega)$ rotates through an angle φ. If the polynomial $f(\lambda)$ has m roots with positive real parts and $n - m$ roots with negative real parts, then

$$\varphi = (n - m)\pi + m(-\pi) = (n - 2m)\pi. \qquad (27.6)$$

Remark. Since the function $u(\omega)$ is even, Mikhailov's curve is symmetric with respect to the u axis, which means that it suffices to construct the part of Mikhailov's curve that

corresponds to values of ω from 0 to $+\infty$. Then formula (27.6) takes the form

$$\varphi = (n-m)\frac{\pi}{2} + m\left(-\frac{\pi}{2}\right) = (n-2m)\frac{\pi}{2}. \quad (27.7)$$

We know that a solution of Eq. (27.1) is stable if and only if all the roots of the characteristic equation $f(\lambda) = 0$ have negative real parts, i.e. m must be zero in (27.7). This leads to the following

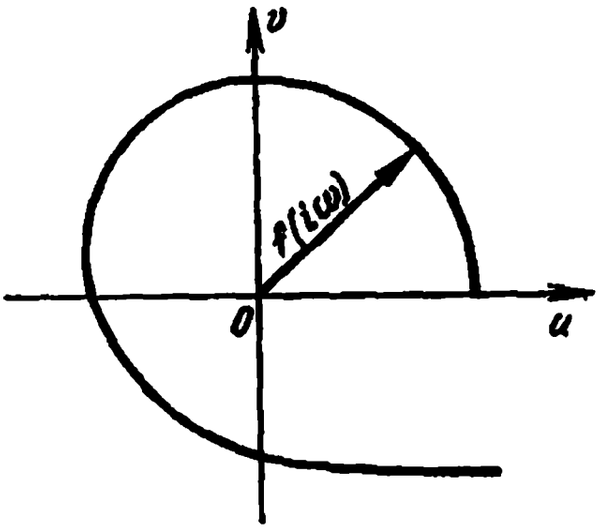


Figure 43

Mikhailov criterion of stability *A solution of Eq.* (27.1) *is stable if and only if* (1) *vector* $f(i\omega)$ *rotates through an angle* $\varphi = n\pi/2$ *as* ω *varies from* 0 *to* $+\infty$, *i.e. the vector makes* $n/4$ *revolutions counterclockwise, and* (2) *the hodograph of vector* $f(i\omega)$ *does not pass through the origin of coordinates as* ω *varies from* 0 to $+\infty$.

In other words, a solution of Eq. (27.1) is stable if and only if Mikhailov's curve passes alternately through n quadrants circling counterclockwise around the origin of coordinates.

The alternate passage through n quadrants means that the curve intersects the coordinate axes. Consequently, the coordinates $u(\omega)$ and $v(\omega)$ of the points of Mikhailov's curve must alternately vanish. This brings us to the *second formulation* of Mikhailov's criterion of stability.

For the zero solution of Eq. (27.1) *to be stable it is necessary (and sufficient if Mikhailov's curve is traversed counterclockwise) that all the roots of the equations* $u(\omega) = 0$, $v(\omega) = 0$ *be real and alternate, i.e. between two roots of one equation there is a root of the other.*

Example. Investigate for stability the zero solution of the equation

$$y^{(IV)} + 2y''' + 3y'' + 2y' + y = 0.$$

Solution. We construct the characteristic polynomial

$$f(\lambda) = \lambda^4 + 2\lambda^3 + 3\lambda^2 + 2\lambda + 1.$$

Moreover,

$$f(i\omega) = \omega^4 - 2i\omega^3 - 3\omega^2 + 2i\omega + 1,$$

$$u(\omega) = w^4 - 3\omega^2 + 1,$$

$$v(\omega) = -2\omega^3 + 2\omega = 2\omega(1 - \omega^2) = 2\omega(1 - \omega)(1 + \omega).$$

Now we vary ω from 0 to $+\infty$ and build Mikhailov's curve (Fig. 44)

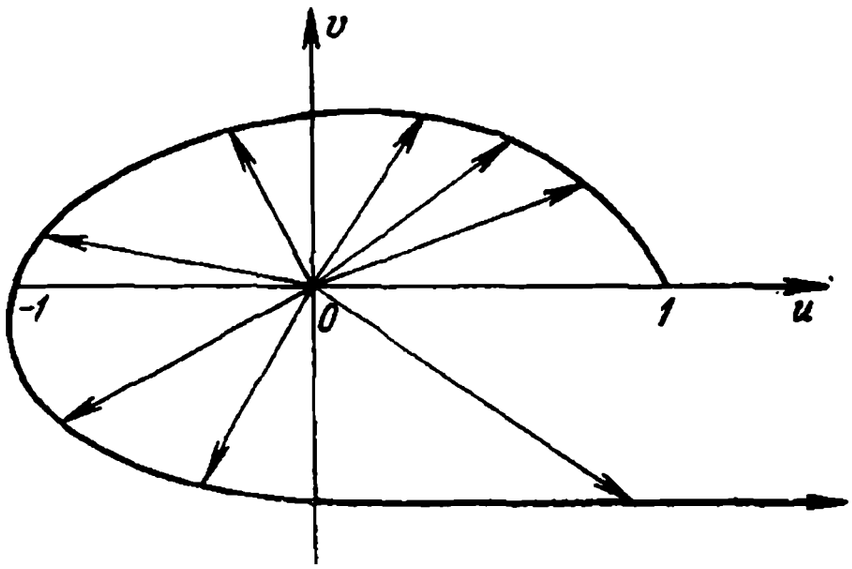


Figure 44

$$\begin{cases} u = u(\omega), \\ v = v(\omega), \end{cases}$$

ω	0	$\sqrt{\frac{3-\sqrt{5}}{2}}$	1	$\sqrt{\frac{3+\sqrt{5}}{2}}$
u	1	0	−1	0
v	0	+	0	−

with $\lim\limits_{\omega \to +\infty} \frac{v}{u} = 0$ (some values of u and v and signs of v are given in the table above). The angle of rotation of the

radius is

$$\varphi - 4\frac{\pi}{2} = (n-2m)\frac{\pi}{2}.$$

This means that $n - 2m = 4$ and $n = 4$; consequently, $m = 0$. Thus, all the curves of the characteristic equation lie in the left half-plane, i.e. the zero solution $y \equiv 0$ is asymptotically stable. We can arrive at the same result via the Liénard-Chipart test, since all the coefficients of the characteristic equation are positive and

$$\Delta_{n-1} = \Delta_3 = \begin{vmatrix} 2 & 1 & 0 \\ 2 & 3 & 2 \\ 0 & 1 & 2 \end{vmatrix} = 4 > 0, \quad \Delta_{n-3} = \Delta_1 = 2 > 0.$$

Investigate for stability the zero solutions of the following equations:

965. $2y^{(IV)} + 4y''' + 3y'' + 3y' + y = 0.$

966. $3y^{(IV)} + 4y''' + 3y'' + 3y' + y = 0.$

967. $y^{(V)} + 5y^{(IV)} + 10y''' + 11y'' + 7y' + 2y = 0.$

968. $y^{(IV)} + 5y''' + 4y'' + 3y' + 2y = 0.$

969. $y^{(V)} + 2y^{(IV)} + 2y''' + 46y'' + 89y' + 260y = 0.$

970. $y^{(V)} + y^{(IV)} + 7y''' + 4y'' + 10y' + 3y = 0.$

971. $y^{(VII)} + 7y^{(VI)} + 23y^{(V)} + 37y^{(IV)} + 56y''' + 36y'' + 12y' + 4y = 0.$

972. $y^{(IV)} + 3y''' + 4y'' + 3y' + y = 0.$

973. $y^{(IV)} + 7y''' + 18y'' + 22y' + 12y = 0.$

974. $y^{(IV)} + 2y''' + 3y'' + 2y' + y = 0.$

975. $y^{(IV)} + 11y''' + 59y'' + 107y' + 60y = 0.$

976. $y^{(IV)} + 5y''' + 18y'' + 53y' + 60y = 0.$

977. $y^{(IV)} + 6y''' + 15y'' + 18y' + 10y = 0.$

978. $y^{(IV)} + 4y''' + 10y'' + 12y' + 5y = 0.$

979. $y''' + 6y'' + 11y' + 6y = 0.$

980. $y^{(IV)} + 7y''' + 12y'' + 23y' + 10y = 0.$

981. $y^{(IV)} + 3y''' + 3y'' + 3y' + 2y = 0.$

982. $y^{(IV)} + 2y''' + 4y'' + 2y' + 5y = 0.$

983. $y^{(IV)} + 2y'' + 8y' + 5y = 0.$

984. $y^{(V)} + 4y^{(IV)} + 5y''' + 2y' + 4y = 0.$

§ 28 Regions of Stability

Suppose we have a linear differential equation with constant real coefficients

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0. \tag{28.1}$$

Its characteristic equation is

$$a_0 z^n + a_1 z^{n-1} + \ldots + a_n = 0. \tag{28.2}$$

There is no need to calculate the roots of the characteristic equation to see whether a solution of Eq. (28.1) is stable or not. It suffices to establish that all the roots lie in the left half-plane. Usually there are two types of this problem.

First type. Assuming that all the coefficients in Eq. (28.1) are given, we wish to establish the stability of a solution for the given values of the coefficients.

Second type. Assuming that only some of the coefficients in Eq. (28.1) are given, we wish to establish the values of the other coefficients for which a solution is stable.

Constructing regions of stability

We start with the characteristic equation

$$a_0 z^n + a_1 z^{n-1} + \ldots + a_n = 0. \tag{28.2}$$

The collection of the values of the coefficients in (28.1) can be taken as a point in an $(n + 1)$-dimensional space R_{n+1}. To each point in R_{n+1} there corresponds a definite set of coefficients $a_0, a_1, \ldots, a_n$ and, hence, a definite set of roots $z_1, z_2, \ldots, z_n$ of the characteristic equation (28.2). If in R_{n+1} there exists a region such that each of its points corresponds to a characteristic equation all of whose roots lie in the left half-plane, the region is said to be a *region of stability* and the hypersurface bounding it the *boundary of the region of stability*. Suppose, for instance, that for the characteristic equation (28.2) all the coefficients except, say, a_1 and a_2 are given. Now let us assume that for certain values of a_1 and a_2 the given equation has k roots lying in the left half of the complex z plane (the plane where the roots lie) and $n - k$ roots in the right half (Fig. 45). In the plane A of parameters a_1 and a_2 there exists a curve that bounds a region (Fig. 46) each point of which defines a polynomial

with k roots lying to the left of the imaginary axis and $n - k$ roots lying to the right of the imaginary axis. We denote this region by $D(k, n-k)$, with k an integer and

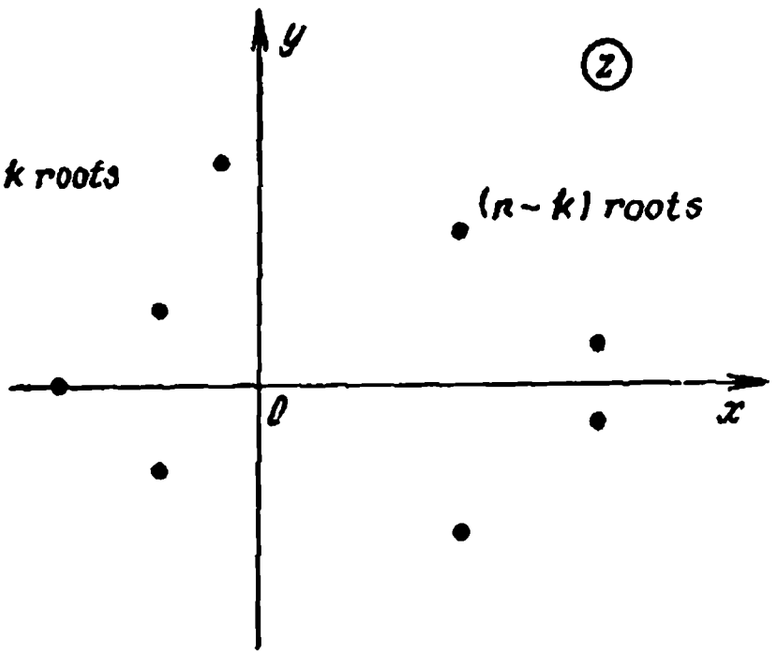


Figure 45

$0 \leqslant k \leqslant n$. For instance, if the characteristic equation is of the third degree, i.e. $n = 3$, then in the coefficients space there are generally four regions:

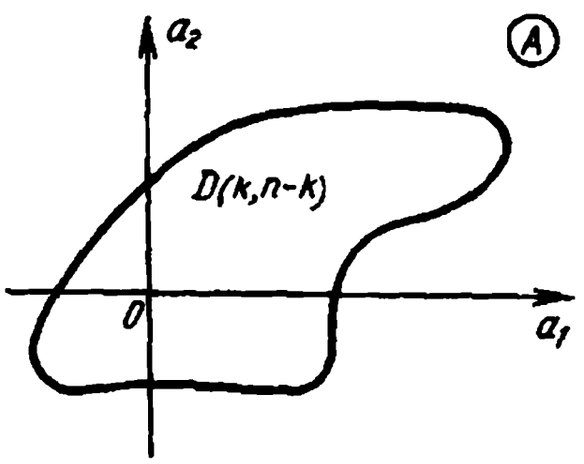


Figure 46

$$D(0, 3),\ D(1, 2),\ D(2, 1),\ D(3, 0).$$

The region $D(3, 0)$ is the region of stability. In some cases, however, some of the regions may not exist, say the region $D(3, 0)$.

The partitioning of the space of coefficients of a characteristic equation into regions that have the same number of roots in the left half of the complex z plane is called *D-partitioning* of the coefficient space. Similarly, we can construct the D-partitioning of the space of any parameters on which the coefficients of a characteristic equation depend.

Suppose that the coefficients in the characteristic equation (28.2) depend on two parameters, ξ and η (these para-

meters may simply be any two coefficients of the equation). We then take the family of the polynomials

$$f(z, \xi, \eta) = \xi P(z) + \eta Q(z) + R(z), \qquad (28.3)$$

where ξ and η are real parameters, and P, Q, and R are known polynomials in z with real coefficients.

The problem can be formulated as follows: In the plane of the parameters ξ and η (the w plane) find a region $D(n, 0)$ such that for any point $(\xi, \eta) \in D(n, 0)$ the polynomial (28.3) has all its roots z in the left half-plane or verify that no such region exists.

The construction of the regions $D(k, n-k)$ is based on the following reasoning: (1) the roots of an algebraic equation are continuous functions of the coefficients of the equation, i.e. a small variation in the coefficients of the polynomial $f(z, \xi, \eta)$ leads to a small variation in the roots, and (2) if a point (ξ, η) lies on the boundary of region $D(k, n-k)$, then at least one root of the polynomial lies on the imaginary axis, i.e. the boundary of the D-partitioning is the image of the imaginary axis of the complex z plane.

Indeed, if, for instance, point (ξ, η) belongs to $D(n, 0)$, the polynomial (28.3) has all its roots in the left half-plane. If (ξ, η) lies outside $D(n, 0)$, at least one root of the polynomial lies in the right half-plane. As we move point (ξ, η) from region $D(n, 0)$ to a neighboring region, all the roots of $f(z, \xi, \eta)$ vary continuously. Since in the process there emerges at least one root in the right half-plane, it must have intersected the imaginary y axis of the complex z plane as point (ξ, η) was moving from region $D(n, 0)$. This happened when point (ξ, η) intersected the boundary of $D(n, 0)$.

Suppose $z = x + iy$ is a root of the polynomial $f(z, \xi, \eta)$. The fact that $f(z, \xi, \eta) = 0$ is equivalent to

$$\begin{cases} \xi u_1(x, y) + \eta u_2(x, y) + u_3(x, y) = 0, \\ \xi v_1(x, y) + \eta v_2(x, y) + v_3(x, y) = 0, \end{cases} \qquad (28.4)$$

where u_1, u_2, u_3 and v_1, v_2, v_3 are the real and imaginary parts of the polynomials P, Q, R, respectively. If the determinant of system (28.4) is nonzero,

$$\Delta = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \neq 0.$$

the system has a unique solution:

$$\begin{cases} \xi = \xi(x, y), \\ \eta = \eta(x, y). \end{cases} \tag{28.5}$$

Equations (28.5) at points where $\Delta \neq 0$ define a unique mapping of the plane of roots of the polynomial $f(z, \xi, \eta)$ onto the plane of parameters ξ and η. The inverse mapping is not unique since to each pair (ξ, η) there may generally correspond n roots. If the system determinant vanishes at point $z_0 = x_0 + iy_0$, the equations in it are either inconsistent or one equation is a corollary of the other. In the latter case there is an entire straight line in the parameter w plane that consists of points (ξ, η) for which $z_0 = x_0 + iy_0$ is a root of $f(z, \xi, \eta)$. Both point (x_0, y_3) and the corresponding straight line are said to be *exceptional.*

Let us find the points in the plane of parameters ξ and η at which the polynomial (28.3) has at least one purely imaginary root $z = iy$. The locus of such points is a curve, whose parametric equation is

$$\begin{cases} \xi = \xi(0, y), \\ \eta = \eta(0, y) \end{cases} \quad (-\infty < y < +\infty) \tag{28.6}$$

and which may be obtained by putting $x = 0$ in Eqs. (28.5), and the exceptional straight lines corresponding to the exceptional points on the y axis (if there are such points). (Note that Eqs. (28.6) give the image of the y axis under the mapping (28.5).) We will call this locus the *L curve*; it partitions the parameter plane into a number of connected regions each of which has the property that for any point (ξ, η) in it the polynomial $f(z, \xi, \eta)$ has the same number of roots in the left half-plane, i.e. is a region of the type $D(k, n - k)$ $0 \leqslant k \leqslant n$). In this way the L curve is the boundary of the sought D-partitioning.

Let us now consider the mapping (28.5) of the root plane onto the parameter plane,

$$\begin{cases} \xi = \xi(x, y), \\ \eta = \eta(x, y). \end{cases}$$

Through point (x_0, y_0) we draw two lines, the horizontal line I and the vertical line II.

Definition. If the sense of rotation from I to II is preserved under the mapping (28.5), we say that the mapping

preserves the orientation at point (x_0, y_0); if this is not the case, the mapping is said *not to preserve the orientation* (Figs. 47 and 48). If

$$I \equiv \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{vmatrix} > 0$$

at point (x_0, y_0), the mapping (28.5) preserves the orientation at the same point. If I is negative, the orientation is not

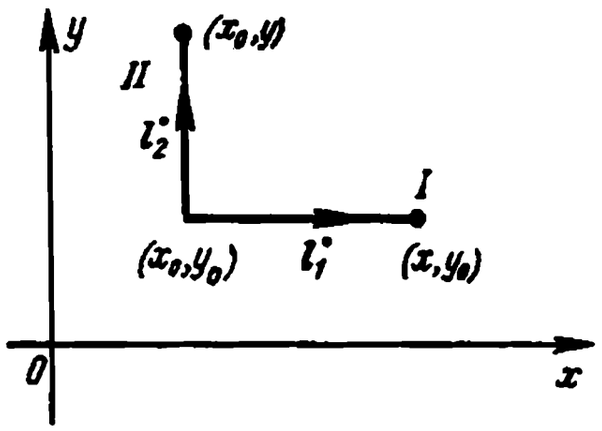


Figure 47

preserved. And if I is zero, the higher-order derivatives determine whether orientation is preserved or not. It can be

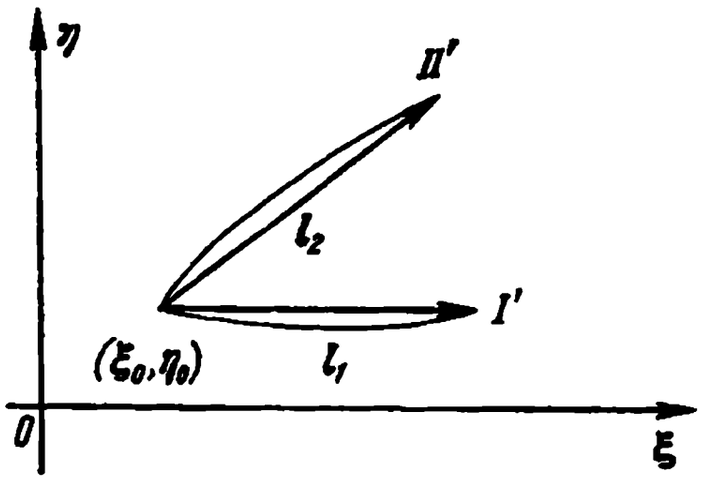


Figure 48

proved (see *Lavrent'ev and Shabat*) that the sign of I coincides with the sign of

$$\Delta = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix},$$

so that if Δ is positive, the mapping of the root plane into the parameter plane preserves the orientation, while if Δ is negative, it does not.

Let us once more consider the partition of the w plane (the parameter plane) into region $D(k, n-k)$, with $k \leqslant n$, and denote the boundary of these regions by L. The positive direction on L is specified by that in which y increases (starting at $y = -\infty$); curve L may consist of several branches and, as the y axis is traversed from $-\infty$ to $+\infty$, these branches may be traversed several times (not greater than n, where n is the degree of the polynomial $f(z, \xi, \eta)$).

Let us take a segment w_1w_2 on L and assume that, as the y axis is traversed from $-\infty$ to $+\infty$, the segment is traversed r times, i.e. r segments $y_1^\mu y_2^\mu$ ($\mu = 1, 2, \ldots, r$) of the y axis correspond to one segment w_1w_2. We put $\varepsilon_\mu = 1$ if the direction of $y_1^\mu y_2^\mu$ coincides with that of the y axis, and $\varepsilon_\mu = -1$ otherwise. We will also assume that $\delta_\mu = 1$ if the determinant Δ is positive on $y_1^\mu y_2^\mu$, and $\delta_\mu = -1$ otherwise. Suppose that while moving continuously along a sufficiently small path a point w crosses the segment w_1w_2 from left to right (Fig. 49). This path corresponds to r paths in the z plane that intersect the segments $y_1^\mu y_2^\mu$ of the y axis. If $\varepsilon_\mu\delta_\mu$ is positive, each path is from the left half-plane to the right half-plane, and the polynomial

$$f(z, \xi, \eta) = \xi P(z) + \eta Q(z) - R(z)$$

acquires one root with a positive real part and loses one with a negative real part; if $\varepsilon_\mu\delta_\mu < 0$, the opposite happens.

Indeed, suppose $\varepsilon_\mu\delta_\mu$ is positive. This can happen in either of two cases: (1) $\varepsilon_\mu = 1$, $\delta_\mu = 1$; (2) $\varepsilon_\mu = -1$, $\delta_\mu = -1$. In the first case the direction of the segment $y_1^\mu y_2^\mu$ of the y axis coincides with the positive direction on this axis ($\varepsilon_\mu = 1$) and orientation is preserved ($\delta_\mu = 1$), i.e. if we cross the arc w_1w_2 in the w plane from left to right, in the z plane we pass from the left half-plane to the right (i.e. the y axis is crossed from left to right, too; see Fig. 50). In the second case the positive direction of the segment $y_1^\mu y_2^\mu$ is opposite to that of the y axis ($\varepsilon_\mu = -1$). Since $\delta_\mu = -1$, the orientation changes to the opposite, and when we go over from the left half of the w plane to the right half, we once more cross the y axis in the z plane from

left to right. The case where $\varepsilon_\mu \delta_\mu$ is negative can be considered in a similar manner.

Thus, when we go over from the left side of the arc $w_1 w_2$ of curve L to the right side, the polynomial $f(z, \xi, \eta)$ loses

$$N = \varepsilon_1 \delta_1 + \varepsilon_2 \delta_2 + \ldots + \varepsilon_r \delta_r$$

roots with negative real parts.

Vyshnegradskii's example. Given the polynomial $f(z) = z^3 + \xi z^2 + \eta z + 1$, find the region $D(3, 0)$.

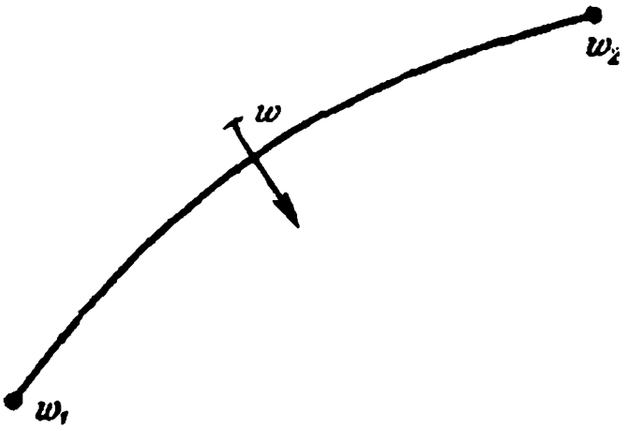


Figure 49

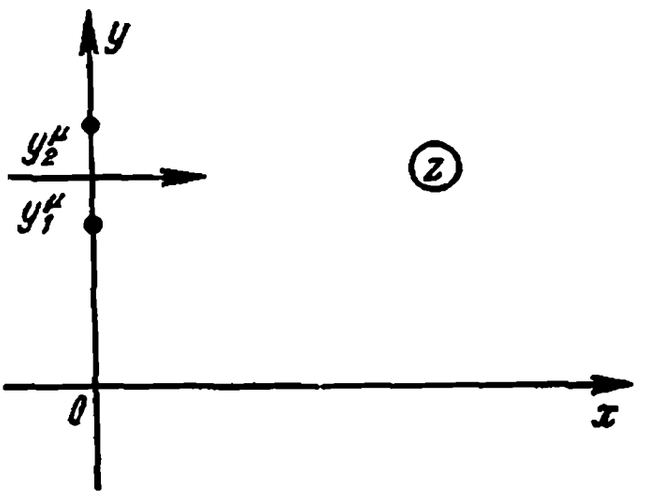


Figure 50

Solution. Putting $z = iy$ and separating the real and imaginary parts we arrive at the parametric equations of the L curve:

$$\xi = \frac{1}{y^2}, \quad \eta = y^2.$$

This is the branch of the hyperbola $\xi\eta = 1$ lying in the first quadrant. As the y axis is traversed from $-\infty$ to $+\infty$, the hyperbola is traversed twice, i.e. $r = 2$; the hyperbola is traversed once (in one direction) when y changes from $-\infty$ to 0. When y continues varying (from 0 to $+\infty$), the hyperbola is traversed the second time but in the opposite direction. Thus, there are two segments of the y axis, $y_1^1 y_2^1$ and $y_1^2 y_2^2$, that correspond to one segment, $w_1 w_2$, of the L curve (Figs. 51 and 52). On the y axis the determinant Δ is equal to $-y^3$ and, consequently, $\delta_1 = 1$ (since $y < 0$ for $\mu = 1$) and $\delta_2 = -1$ (since $y > 0$ for $\mu = 2$). As point w crosses $w_1 w_2$ from left to right, we loose

$$N = \varepsilon_1 \delta_1 + \varepsilon_2 \delta_2 = 2$$

roots with negative real parts. At the origin of coordinates $\xi = \eta = 0$ the polynomial $f(z)$ takes the form $z^3 + 1$ and has the following roots: $z_1 = -1$ and $z_{2,3} = (1 \pm i\sqrt{3})/2$; therefore the region under hyperbola is D (1, 2). The region above the hyperbola is D (3, 0). Indeed, when we pass from this region to D (1, 2), the polynomial $f(z)$ loses two roots with negative real parts and becomes a polynomial with

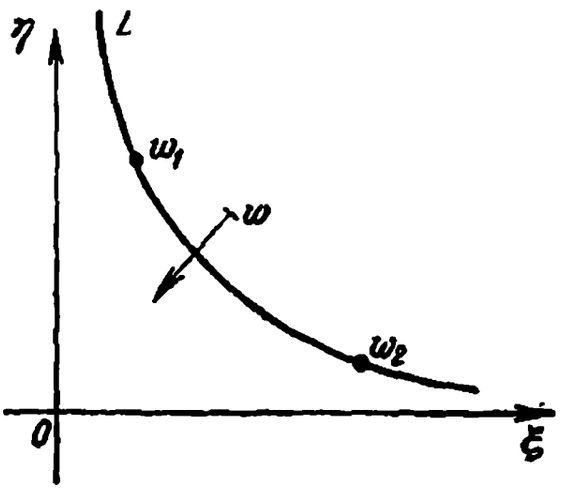


Figure 51

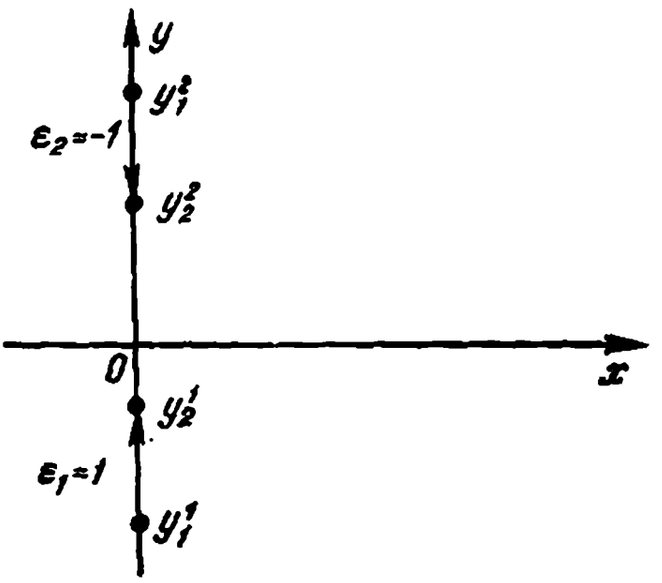


Figure 52

only one root with a negative real part (see above). Therefore, there were three roots with negative real parts above the hyperbola (Fig. 53). To verify this we may take the point $\xi = \eta = 3$, at which the polynomial takes the form

$$z^3 + 3z^2 + 3z + 1$$

and has the triple root $z = -1$.

Summarizing, we see four steps in constructing the D regions:

(1) We put $z = iy$ in the polynomial $f(z, \xi, \eta)$, separate the real and imaginary parts, and nullify them,

$$\begin{cases} \xi u_1(y) + \eta u_2(y) + u_3(y) = 0, \\ \xi v_1(y) + \eta v_2(y) + v_3(y) = 0. \end{cases} \tag{28.7}$$

Solving (28.7) for ξ and η, we find

$$\begin{cases} \xi = \xi(y), \\ \eta = \eta(y) \end{cases}$$

which is the L curve.

(2) We construct the L curve in the parameter plane and vary y from $-\infty$ to $+\infty$, and if in Eqs. (28.7) ξ is the first parameter and η the second, we select a right-handed coordinate system (ξ, η) in the plane with the L curve.

If for some value of y the determinant of system (28.7) and the determinants

$$\Delta_\xi = \begin{vmatrix} -u_3 & u_2 \\ -v_3 & u_2 \end{vmatrix} \quad \text{and} \quad \Delta_\eta = \begin{vmatrix} u_1 & -u_3 \\ v_1 & -v_3 \end{vmatrix}$$

vanish, for this value of y one equation in (28.7) is a corollary of the other and (for this value of y) we have a straight

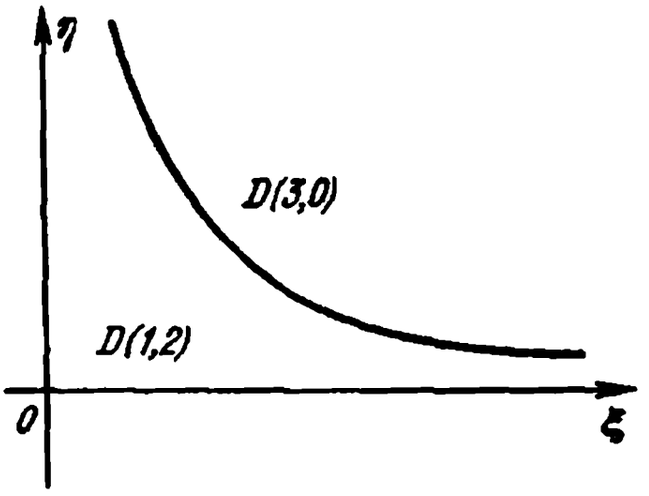


Figure 53

line (an exceptional straight line) in the $\xi O \eta$ plane instead of a point. We include it in the boundary of the D-partitioning.

If the leading coefficient in the characteristic equation depends on ξ and η, nullifying this coefficient we arrive at the equation of another exceptional straight line, which corresponds to $y = \infty$.

Finally, if the determinant of system (28.7) is zero, the boundary of the D-partitioning consists of exceptional straight lines only.

(3) We isolate the connected domains into which the L curve partitions the parameter plane. These are the regions $D(k, n-k)$ $(0 \leqslant k \leqslant n)$.

(4) Finally, we determine the nature of these regions, i.e find k and $n-k$. To this end we take one point (ξ_0, η_0) in each of the regions $D(k, n-k)$ and investigate the polynomial $f(z, \xi_0, \eta_0)$ with numerical coefficients for stability

via the Routh-Hurwitz or Mikhailov's criterion (see §§ 26 and 27).

Construct the D regions for the following polynomials:

985. $z^3 + \xi z^2 + \eta z + 6$. 986. $z^4 + \xi z^3 + \eta z^2 + 4z + 1$.
987. $z^3 + \xi z^2 + 11z + \eta$. 988. $z^3 + (z^2 + 2)\xi + \eta z - 4$.
989. $z^4 + 2z^3 + \xi z^2 + z + \eta$.
990. $z^3 + 3z^2 + \xi z + \eta$.
991. $z^3 + \xi z^2 + (z + 1)\eta + 1$.
992. $z^3 + \eta z^2 + \xi z + 6$.
993. $z^3 + 2z^2 + \xi(z - 1) + \eta$.
994. $z^3 + \xi(z^2 + z) + z + 2\eta$.
995. $\xi z^3 + 3z^2 + \eta z + 1$.
996. $\xi(z^3 + z^2) + \eta(z^2 + 1) + 2z$.
997. $\xi(z^3 - z) + \eta(z^2 + z - 1) + 1$.

§ 29 Stability of Solutions of Difference Equations

Solving homogeneous linear difference equations with constant coefficients

Suppose we have the difference equation of order k

$$f(n + k) + a_1 f(n + k - 1) + \ldots + a_k f(n) = 0, \quad (29.1)$$

where $a_k \neq 0$, $f(n)$ is the unknown function of an integral argument, and $a_1, \ldots, a_k$ are real constants.

To find the nonzero solutions of Eq. (29.1) we construct the characteristic equation

$$\lambda^k + a_1\lambda^{k-1} + \ldots + a_{k-1}\lambda + a_k = 0. \quad (29.2)$$

Suppose $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the roots of Eq. (29.2). Four cases are possible:

(1) $\lambda_1, \lambda_2, \ldots, \lambda_k$ are real and distinct. The general solution of Eq. (29.1) is

$$f(n) = C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_k\lambda_k^n, \quad (29.3)$$

where $C_1, C_2, \ldots, C_k$ are arbitrary constants, which can be found if we know the initial conditions

$$f(0) = f_0, f(1) = f_1, \ldots, f(k - 1) = f_{k-1}.$$

(2) The roots of the characteristic equation are real but there are multiple roots among them. For instance, suppose

that $\lambda_1 = \lambda_2 = \ldots = \lambda_j = \tilde{\lambda}$, i.e. $\tilde{\lambda}$ is a j-tuple root of Eq. (39.2), while all the other roots are distinct (simple). The general solution of Eq. (29.1) is

$$f(n) = C_1\tilde{\lambda}^n + C_2 n\tilde{\lambda}^n + \ldots + C_j n^{j-1}\tilde{\lambda}^n - C_{j+1}\lambda_{j+1}^n + \ldots + C_k\lambda_k^n. \quad (29.4)$$

(3) There are simple complex-valued roots among the roots of the characteristic equation (29.2). Suppose, for instance, that

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta,$$
$$\lambda_3 = \gamma + i\delta, \quad \lambda_4 = \gamma - i\delta,$$

while all the other roots are real and simple. The general solution (29.1) has the form

$$f(n) = C_1 |\lambda_1|^n \cos(n \arg \lambda_1) + C_2 |\lambda_1|^n \sin(n \arg \lambda_1) + C_3 |\lambda_3|^n \cos(n \arg \lambda_3) + C_4 |\lambda_3|^n \sin(n \arg \lambda_3) + C_5\lambda_5^n + \ldots + C_k\lambda_k^n. \quad (29.5)$$

(4) If $\lambda_1 = \alpha + i\beta$ is a j-tuple root of Eq. (29.2) ($j \leqslant k/2$), then $\lambda_2 = \alpha - i\beta$ is also a j-tuple root of this equation, and the general solution of Eq. (29.1) is

$$f(n) = (C_1 + C_2 n + \ldots + C_j n^{j-1}) |\lambda_1|^n \cos(n \arg \lambda_1) + (C_{j+1} + C_{j+2} n + \ldots + C_{2j} n^{j-1}) |\lambda_1|^n \sin(n \arg \lambda_1) + C_{2j+1}\lambda_{2j+1}^n + \ldots + C_k\lambda_k^n. \quad (29.6)$$

Remark. The root $\lambda = 0$ corresponds to the zero solution $f(n) \equiv 0$.

Example 1. Find the general solution of the equation

$$f(n+2) + 4f(n+1) + f(n) = 0.$$

Solution. The characteristic equation is

$$\lambda^2 + 4\lambda + 1 = 0.$$

Its roots are $\lambda_1 = -2 - \sqrt{3}$ and $\lambda_2 = -2 + \sqrt{3}$, which are real and distinct. Consequently,

$$f(n) = C_1(-2 - \sqrt{3})^n + C_2(-2 + \sqrt{3})^n.$$

Example 2. Find the general solution of the equation

$$f(n+3) - 3f(n+2) + 3f(n+1) - f(n) = 0.$$

Solution. The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0, \text{ or } (\lambda - 1)^3 = 0.$$

This yields $\lambda_1 = \lambda_2 = \lambda_3 = 1$. The general solution is

$$f(n) = (C_1 + C_2 n + C_3 n^2) \times 1^n = C_1 + C_2 n + C_3 n^2.$$

Example 3. Find the general solution of the equation

$$f(n+2) - 2f(n+1) + 2f(n) = 0.$$

Solution. The characteristic equation

$$\lambda^2 - 2\lambda + 2 = 0$$

has only simple complex-valued roots,

$$\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i.$$

We find that

$$|1 \pm i| = \sqrt{2}, \ \arg(1 + i) = \frac{\pi}{4}.$$

The general solution is

$$f(n) = C_1 2^{\frac{n}{2}} \cos\frac{n\pi}{4} + C_2 2^{\frac{n}{2}} \sin\frac{n\pi}{4}$$

$$= 2^{\frac{n}{2}} \left(C_1 \cos\frac{n\pi}{4} + C_2 \sin\frac{n\pi}{4}\right).$$

Example 4. Find the general solution of

$$f(n+4) + 2f(n+3) + 4f(n+2) - 2f(n+1) - 5f(n) = 0.$$

Solution. The characteristic equation is

$$\lambda^4 + 2\lambda^3 + 4\lambda^2 - 2\lambda - 5 = 0.$$

We write it in the following form:

$$(\lambda^2 - 1)(\lambda^2 + 2\lambda + 5) = 0.$$

The roots are

$$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = -1 + 2i, \quad \lambda_4 = -1 - 2i.$$

We see that

$$|-1 \pm 2i| = \sqrt{5}, \quad \arg(-1 + 2i) = \pi - \arctan 2.$$

The general solution is

$$f(n) = C_1 + C_2(-1)^n + [C_3 \cos n(\pi - \arctan 2) + C_4 \sin n(\pi - \arctan 2)]\, 5^{n/2},$$

or

$$f(n) = C_1 + C_2(-1)^n + (-1)^n 5^{n/2} \times [C_3 \cos(n \arctan 2) - C_4 \sin(n \arctan 2)].$$

Solve the following difference equations:

998. $3f(n+2) - 2f(n+1) - 8f(n) = 0$.

999. $f(n+3) + 3f(n+2) + 3f(n+1) - f(n) = 0$, $f(0) = 1$, $f(1) = 2$, $f(2) = 3$.

1000. $4f(n+2) - 8f(n+1) + 5f(n) = 0$.

1001. $f(n+3) - 8f(n) = 0$.

1002. $f(n+4) - f(n+2) + 2f(n+1) + 2f(n) = 0$.

Solving nonhomogeneous linear difference equations with constant coefficients

Suppose we are dealing with the nonhomogeneous linear difference equation of order k

$$(n+k) + a_1 f(n+k-1) + \ldots + a_k f(n) = g(n) \quad (29.7)$$

with constant coefficients $a_1, \ldots, a_k$. The general solution of this equation can be expressed as the sum of any particular solution of the equation and the general solution of the complementary equation (29.1).

(1) Let the right-hand side of Eq. (29.7) be

$$g(n) = r^n u(n),$$

where $u(n)$ is an mth degree polynomial in n, and r is a real number. If r is not a root of the characteristic equation (29.2), we can look for a particular solution in the form

$$\tilde{f}(n) = r^n \tilde{u}(n),$$

where $\tilde{u}(n)$ is an mth degree polynomial. But if r is an j-tuple root of Eq. (29.2), then $\tilde{u}(n)$ is an $(m+j)$ th degree polynomial.

(2) If $g(n)$ is of the form

$$g(n) = u(n) \sin \alpha n \text{ or } g(n) = u(n) \cos \alpha n$$

we look for a particular solution in the form

$$\tilde{f}(n) = \tilde{u}(n) \sin \alpha n + \tilde{\tilde{u}}(n) \cos \alpha n.$$

(3) If $g(n) = u(n) \sinh \alpha n$ or $g(n) = u(n) \cosh \alpha n$, we look for a particular solution in the form

$$\tilde{f}(n) = \tilde{u}(n) \sinh \alpha n + \tilde{\tilde{u}}(n) \cosh \alpha n.$$

Both here and in item (2) $\tilde{u}(n)$ and $\tilde{\tilde{u}}(n)$ are polynomials whose degrees are found by the same rules as in item (1).

Example 5. Find the general solution of the equation

$$f(n+2) - 4f(n+1) + 3f(n) = 2^n(n+1). \quad (29.8)$$

Solution. The roots of the characteristic equation $\lambda^2 - 4\lambda + 3 = 0$ are $\lambda_1 = 3$ and $\lambda_2 = 1$. The general solution of the complementary equation is

$$f(n) = 3^n C_1 + C_2.$$

Since 2 is not a root of the characteristic equation, we look for a partial solution of the nonhomogeneous equation in the form

$$\tilde{f}(n) = 2^n (An + B), \quad (29.9)$$

where A and B are yet to be found. Substituting (29.9) into (29.8), we obtain

$$2^{n+2}(An + 2A + B) - 4 \times 2^{n+1}(An + A + B) + 3 \times 2^n (An + B) = 2^n (n + 1),$$

or

$$4(An + 2A + B) - 8(An + A + B) + 3(An + B) = n + 1.$$

This yields

$$4A - 8A + 3A = 1,$$

$$8A + 4B - 8A - 8B + 3B = 1,$$

so that $A = -1$ and $B = -1$. Thus, the partial solution we have found is

$$\tilde{f}(n) = -2^n(n+1);$$

the general solution is

$$\tilde{f}(n) = 3^n C_1 + C_2 - 2^n(n+1).$$

Find the general solutions of the following nonhomogeneous linear difference equations:

1003. $f(n+2) - 2f(n+1) - f(n) = n.$

1004. $f(n+2) + 2f(n+1) + f(n) = 3^n \times 32,$ $f(0) = 0,\ f(1) = 0.$

1005. $f(n+2) + f(n) = \sin 2n,\ f(0) = 0,\ f(1) = 1.$

1006. $f(n+3) - 3f(n+2) + 3f(n+1) - f(n) = e^n.$

1007. $f(n+3) + 8f(n) = 2^n.$

Stability of solutions of difference equations

A solution $f^*(n)$ of a kth order difference equation that satisfies the initial conditions

$$f^*(0) = f_0^*,\ f^*(1) = f_1^*,\ \ldots,\ f^*(k-1) = f_{k-1}^*$$

is said to be *stable* if for every positive ε there is a positive $\delta(\varepsilon)$ such that for any solution $f(n)$ of Eq. (29.1) that satisfies the initial conditions

$$f(0) = f_0,\ f(1) = f_1,\ \ldots,\ f(k-1) = f_{k-1}$$

the inequalities

$$|f_0 - f_0^*| < \delta,\ |f_1 - f_1^*| < \delta,\ \ldots,\ |f_{k-1} - f_{k-1}^*| < \delta$$

imply $|f(n) - f^*(n)| < \varepsilon$ for any $n \geqslant 0$. If for an arbitrarily small positive $\delta(\varepsilon)$ the inequality $|f(n) - f^*(n)| < \varepsilon$ is not met for a solution $f(n)$, the solution $f^*(n)$ is said to be *unstable*. If in addition to $|f(n) - f^*(n)| < \varepsilon$ we find that

$$\lim_{n\to\infty} [f(n) - f^*(n)] = 0.$$

the solution $f^*(n)$ is said to be *asymptotically stable*.

The investigation for stability of the solution $f^*(n)$

of the nonhomogeneous linear difference equation

$$f(n+k)+a_1f(n+k-1)+\ldots+a_kf(n)=g(n)$$

can be reduced, via the substitution $\varphi(n)=f(n)-f^*(n)$, to the investigation for stability of the zero solution of the homogeneous equation

$$\varphi(n+k)+a_1\varphi(n+k-1)+\ldots+a_k\varphi(n)=0.$$

In what follows we will restict ourselves to zero solutions of homogeneous equations.

Example 6. Starting from the definition of a stable solution of a difference equation, investigate for stability the solution of the difference equation

$$2f(n+2)-2f(n+1)+f(n)=0, \qquad (29.10)$$

satisfying the initial conditions $f(0)=0$, $f(1)=0$.

Solution. The solution of the given equation satisfying the initial conditions $f(0)=1$, $f(1)=0$ is

$$f(n)\equiv 0,$$

since (29.10) yields

$$f(n+2)=f(n+1)-\frac{1}{2}f(n).$$

Every solution of this equation that satisfies the conditions $f(0)=f_0$, $f(1)=f_1$ has the form

$$f^*(n)=\frac{1}{2^{n/2}}\left[f_0\cos\frac{n\pi}{4}+(f_1-f_0)\sin\frac{n\pi}{4}\right].$$

Let us take an arbitrary $\varepsilon>0$ and show that there exists a positive $\delta(\varepsilon)$ such that for $|f_0-0|<\delta$ and $|f_1-0|<\delta$ we have

$$|0-f^*(n)|=\frac{1}{2^{n/2}}\left|f_0\cos\frac{n\pi}{4}+(f_1-f_0)\sin\frac{n\pi}{4}\right|<\varepsilon$$

for all $n\geqslant 0$. According to the definition, this would mean that the zero solution $f^*(n)\equiv 0$ is stable. It is obvious that

$$\frac{\left|f_0\cos\frac{n\pi}{4}+(f_1-f_0)\sin\frac{n\pi}{4}\right|}{2^{n/2}}\leqslant\frac{|f_0|+|f_1-f_0|}{2^{n/2}}$$

$$\leqslant|f_0|+|f_1-f_0|\leqslant|f_0|+|f_1|+|f_0|\leqslant 2(|f_0|+|f_1|)$$

for all $n \geqslant 0$. Whence, if $|f_0| + |f_1|$ is less than $\varepsilon/2$, then certainly $|0 - f^*(n)|$ is less than ε for all $n \geqslant 0$. Consequently, if, for instance, we take $\delta(\varepsilon) = \varepsilon/4$, then for $|f_0| < \delta$ and $|f_1| < \delta$ the inequality $|0 - f^*(n)| < \varepsilon$ is satisfied for all $n \geqslant 0$, so that the zero solution of the given equation is indeed stable. This stability is asymptotic, since

$$\lim_{n\to\infty} [0 - f^*(n)] = -\lim_{n\to\infty} \frac{f_0 \cos \frac{n\pi}{4} + (f_1 - f_0) \sin \frac{n\pi}{4}}{2^{n/2}} = 0.$$

Using the definition of stability of solution, investigate for stability the zero solutions of the following difference equations:

1008. $8f(n+2) + 2f(n+1) - f(n) = 0.$

1009. $f(n+2) + f(n) = 0.$

1010. $4f(n+2) - 4f(n+1) + f(n) = 0.$

1011. $f(n+2) - 6f(n+1) - 7f(n) = 0.$

To investigate the zero solution $f(n)$ of Eq. (29.1) for stability the following general rule is usually used:

(1) If all the roots of the characteristic equation (29.2) are less than unity in absolute value, the solution $f(n) \equiv 0$ of Eq. (29.1) is asymptotically stable.

(2) If at least one root of the characteristic equation is greater than unity in absolute value, the solution $f(n) \equiv 0$ is unstable.

(3) If the characteristic equation has simple roots that are equal to unity in absolute value while the other roots, if they exist, are less than unity in absolute value, the solution $f(n) \equiv 0$ is stable but not asymptotically.

(4) If the characteristic equation has at least one multiple root equal to unity in absolute value, the solution $f(n)$ is unstable.

This rule reduces the problem of determining the stability of the zero solution of Eq. (29.1) to that of determining whether the absolute values of the roots of Eq. (29.2) are greater than, equal to, or less than unity.

Example 7. Investigate for stability the zero solution $f(n) \equiv 0$ of the equation

$$2f(n+2) - 2f(n+1) + f(n) = 0.$$

Solution. We construct the characteristic equation

$$2\lambda^2 - 2\lambda + 1 = 0.$$

Its roots are $\lambda_{1,2} = (1 \pm i)/2$. We have

$$|\lambda_{1,2}| = \left|\frac{1 \pm i}{2}\right| = \frac{1}{\sqrt{2}} < 1.$$

Therefore, the solution $f(n) \equiv 0$ of the given equation is asymptotically stable.

Example 8. Investigate for stability the zero solution of the equation

$$f(n+2) - 2f(n+1) + 5f(n) = 0.$$

Solution. The characteristic equation

$$\lambda^2 - 2\lambda + 5 = 0$$

has two roots:

$$\lambda_1 = 1 + 2i, \quad \lambda_2 = 1 - 2i.$$

We have

$$|\lambda_1| = |\lambda_2| = |1 \pm 2i| = \sqrt{5} > 1.$$

Both roots are greater than unity in absolute value, which means that the solution $f(n) \equiv 0$ is unstable. The problem is solved.

We know that the function

$$\lambda = \frac{w+1}{w-1}$$

maps the interior of the unit circle in the complex λ plane onto the left half of the complex w plane. The roots of the characteristic equation (29.2) that lie in the interior of the unit circle, $|\lambda| < 1$ (i.e. the roots that are less than unity in absolute value), correspond to the roots of the transformed equation

$$(w+1)^k + a_1(w+1)^{k-1}(w-1) + \ldots - a_k(w-1)^k = 0,$$

or

$$b_0 w^k + b_1 w^{k-1} + \ldots + b_n = 0, \qquad (29.11)$$

that lie in the left half of the complex w plane. The location of the roots of Eq. (29.11) can be determined by the Routh-Hurwitz or Mikhailov's criterion.

Example 9. Find the necessary and sufficient conditions for the roots of the characteristic equation

$$\lambda^2 + a_1\lambda + a_2 = 0 \qquad (29.12)$$

to lie in the interior of the unit circle, $|\lambda| < 1$.

Solution. Let us put $\lambda = (w + 1)/(w - 1)$. Then Eq. (29.12) takes the form

$$(w + 1)^2 + a_1 (w + 1)(w - 1) + a_2 (w - 1)^2 = 0,$$

or

$$(1 + a_1 + a_2)w^2 + (2 - 2a_2)w + (1 - a_1 + a_2) = 0. \quad (29.13)$$

We apply the Routh-Hurwitz criterion (see § 26) to the polynomial (29.13). Hurwitz's matrix is

$$\begin{pmatrix} 2 - 2a_2 & 1 + a_1 + a_2 \\ 0 & 1 - a_1 + a_2 \end{pmatrix}.$$

The principal diagonal minors of Hurwitz's matrix are

$$\Delta_1 = 2 - 2a_2, \quad \Delta_2 = (2 - 2a_2)(1 - a_1 + a_2).$$

By the above-mentioned criterion,

$$1 + a_1 + a_2 > 0, \quad 1 - a_2 > 0, \quad 1 - a_1 + a_2 > 0. \quad (29.14)$$

Thus, the characteristic equation (29.12) has roots inside the circle $|\lambda| < 1$ if and only if the conditions (29.14) are met.

Corollary. The homogeneous linear difference equation of second order with constant coefficients,

$$f(n + 2) + a_1 f(n + 1) + a_2 f(n) = 0,$$

has an asymptotically stable zero equation $f(n) \equiv 0$ if and only if its coefficients satisfy conditions (29.14).

Example 10. Investigate for stability the zero solution $f(n) \equiv 0$ of the equation

$$2f(n + 2) - 2f(n + 1) + f(n) = 0.$$

Solution. We rewrite this equation thus:

$$f(n + 2) - f(n + 1) + \frac{1}{2} f(n) = 0.$$

Here $a_1 = -1$ and $a_2 = 0.5$ Whence

$$1 + a_1 + a_2 = 0.5 > 0,$$
$$1 - a_2 = 0.5 > 0,$$
$$1 - a_1 + a_2 = 2.5 > 0.$$

The conditions (29.14) of the Routh-Hurwitz criterion are met. This means that the solution $f(n) \equiv 0$ is asymptotically stable.

Example 11. Investigate for stability the zero solution of the equation

$$f(n+2) + f(n+1) + 2f(n) = 0.$$

Solution. Here $a_1 = 1$ and $a_2 = 2$. We have

$$1 + a_1 + a_2 = 4 > 0,$$
$$1 - a_2 = -1 < 0.$$

The zero solution is unstable.

Find the necessary and sufficient conditions for the zero solutions of the following difference equations to be asymptotically stable:

1012. $a_0 f(n+3) + a_1 f(n+2) + a_2 f(n-1) + a_3 f(n) = 0.$

1013. $f(n+4) + pf(n+2) + qf(n) = 0.$

1014. $f(n+5) + pf(n) = 0.$

1015. $af(n+5) - bf(n) = 0,\ a \neq 0,\ b > 0.$

Using the Routh-Hurwitz criterion, investigate for stability the zero solutions of the following difference equations:

1016. $11f(n+4) - 8f(n+3) + 8f(n+2) - 4f(n+1) + f(n) = 0.$

1017. $f(n+4) + f(n+3) + f(n) = 0.$

1018. $12f(n+4) - 3f(n+3) + 2f(n+2) + 2f(n+1) - 2f(n) = 0.$

1019. $7f(n+4) - 4f(n+3) + 30f(n+2) - 4f(n+1) + 3f(n) = 0.$

1020. $f(n+5) - f(n+1) + f(n) = 0.$

1021. $f(n+5) - f(n+2) - f(n) = 0.$

1022. $f(n+5) + f(n+1) - f(n) = 0.$

Answers

2. $x=\frac{20}{17}$, $y=-\frac{36}{17}$. 3. $x=\frac{-b}{a^2+b^2}$, $y=\frac{-a}{a^2+b^2}$.

4. No real-valued solution exists. 5. $\frac{2(a^2-b^2)}{(a^2+b^2)^2}$.

7. $x=\frac{u^2+v^2-u}{(1-u)^2+v^2}$, $y=-\frac{v}{(1-u)^2+v^2}$.

8. $z_1=0$, $z_2=1$, $z_3=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, $z_4=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$.

9. (a) $\rho=5$, $\varphi=\arctan\frac{3}{4}$; (b) $\rho=4$, $\varphi=\frac{2}{3}\pi$; (c) $\rho=5\sqrt{2}$, $\varphi=\arctan\frac{1}{7}-\pi$; (d) $\rho=1$, $\varphi=\frac{4}{5}\pi$; (e) $\rho=5$, $\varphi=-\arctan\frac{3}{4}$; (f) $\rho=1$, $\varphi=2\pi-\alpha$.

10. (a) $2(\cos\pi+i\sin\pi)$; (b) $2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$; (c) $2\left(\cos\frac{3}{4}\pi+\right.$ $\left.+i\sin\frac{3}{4}\pi\right)$; (d) $\sqrt{2(1-\sin\alpha)}\left[\cos\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{2}+\frac{\alpha}{2}\right)\right]$; (e) $1(\cos\alpha+i\sin\alpha)$; (f) $2e^{i\pi}$; (g) $1\times e^{i\frac{\pi}{2}}$; (h) $1\times e^{-i\frac{\pi}{2}}$; (i) $2e^{-\frac{2\pi}{3}i}$; (j) $1\times e^{\left(\alpha-\frac{\pi}{2}\right)i}$; (k) $\sqrt{34}e^{i\arctan\frac{3}{5}}$.

11. *Hint.* Since $x^2-2\lambda x\cos\alpha+\lambda^2=[x-\lambda(\cos\alpha+i\sin\alpha)]\times$ $\times[x-\lambda(\cos\alpha-i\sin\alpha)]$, we must show that $f[\lambda(\cos\alpha\pm i\sin\alpha)]=$ $=0$.

12. (a) $-2^{19}(1+i\sqrt{3})$; (b) $2^{10}(1+i)$; (c) 1728; (d) 1.

13. *Hint.* Use the relation

$$\frac{1+i\tan\alpha}{1-i\tan\alpha}=\frac{\cos\alpha+i\sin\alpha}{\cos(-\alpha)+i\sin(-\alpha)}$$

and apply de Moivre's formula to the numerator and denominator.

15. (a) $3\cos^2\varphi\sin\varphi-\sin^3\varphi$; (b) $\cos^3\varphi-3\cos\varphi\sin^2\varphi$; (c) $4\sin\varphi\cos^3\varphi-4\cos\varphi\sin^3\varphi$; (d) $\cos^4\varphi-6\cos^2\varphi\sin^2\varphi+\sin^4\varphi$; (e) $5\sin\varphi\cos^4\varphi-10\cos^2\varphi\sin^3\varphi+\sin^5\varphi$; (f) $\cos^5\varphi-10\cos^3\varphi\times\times\sin^2\varphi+5\cos\varphi\sin^4\varphi$.

16. (a) $\pm\dfrac{1\pm i}{\sqrt{2}}$; (b) $\pm\dfrac{1}{\sqrt{2}}(1+i)$; (c) $\dfrac{1}{2}(\pm\sqrt{3}+i)$, $-i$; (d) $\pm\left(\cos\dfrac{\pi}{8}-i\sin\dfrac{\pi}{8}\right)$, $\pm\left(\cos\dfrac{3}{8}\pi+i\sin\dfrac{3}{8}\pi\right)$.

17. (a) ±1; $\pm i$; (b) $\dfrac{\sqrt[3]{4}}{2}(1+i)$, $\sqrt[6]{2}\left(-\cos\dfrac{\pi}{12}+i\sin\dfrac{\pi}{12}\right)$, $\sqrt[6]{12}\left(\sin\dfrac{\pi}{12}-i\cos\dfrac{\pi}{12}\right)$; (c) $\pm(\sqrt{3}-i)$.

18. $\sqrt[10]{2}(\cos 6^\circ+i\sin 6^\circ)$; $\sqrt[10]{2}(\cos 78^\circ+i\sin 78^\circ)$; $\sqrt[10]{2}(\cos 150^\circ+$ $+i\sin 150^\circ)$; $\sqrt[10]{2}(\cos 222^\circ+i\sin 222^\circ)$; $\sqrt[10]{2}(\cos 294^\circ+i\sin 294^\circ)$.

19. (a) The entire complex plane without a circle of radius 2 centered at the origin of coordinates. (b) The circle of radius $r=1$ centered at the origin of coordinates without the center of the circle. (c) The entire complex plane without a circle of radius $r=1/2$ centered at the origin of coordinates.

20. (a) The circumference of a circle of radius $r=8$ centered at point $z=5i$. (b) A circle of radius $r=4$ centered at point $z=1+i$ and its boundary.

21. (a) The part of an annulus bounded by two rays, $\arg z=\pi/4$ and $\arg z=\pi/2$, and the circumferences of two circles with radii $r=1$ and $r=2$ centered at the point $z=-i$. (b) The part of the annulus bounded by two rays $\arg z=\pi/8$ and $\arg z=4\pi/3$ and the circumferences of two circles of radii $r=2$ and $r=3$ centered at point $z=0$. The set also includes the part of the ray $\arg z=4\pi/3$ between the above-mentioned circumferences and the part of the circumference of radius 3 between the above-mentioned rays.

22. (a) The right half-plane including the y axis; (b) the strip between the straight lines $y=0$ and $y=1$ including the straight lines.

23. (a) A concentric annulus bounded by the circumferences of circles with radii $R_1=1$ and $R_2=2$ centered at point $z_0=-(2+i)$; both circumferences belong to the set. (b) The part of the plane that lies below the straight line $y=x$. (c) The strip between the straight lines $x=1$ and $x=2$.

24. The interior of the unit circle.

25. (a) The exterior of the parabola $y=x^2/4-1$; (b) the real semiaxis including point (0, 0).

26. The interior of the hyperbola $xy=-1/2$.

27. The annular region between the ellipses $x^2/4+y^2/3=1$ and $x^2/16+y^2/15=1$ including the two ellipses.

28. The interior of the circle $x^2+(y-1)^2=1$; (b) the region lying between the circles

$$(x-1)^2+(y-1)^2=2 \text{ and } (x-2)^2+(y-2)^2=8.$$

29. The straight line $x=-2$.

30. The straight line $y=2$.

31. (a) The hyperbola $xy = 1$; (b) the hyperbola $x^2 - y^2 = 1$; (c) the circle $x^2 + (y + 1)^2 = 1$.

32. (a) The circle $(x - 1/2)^2 + y^2 = 1/4$; (b) the hyperbola $xy = -1$.

33. The hyperbola $x^2 - y^2 = 1/2$.

34. The circle $(x + 1)^2 + (y - 1/2)^2 = 9/4$.

35. (a) The ellipse $x^2/3 + y^2/4 = 1$; (b) the ray on the y axis from -1 to $-\infty$.

36. (a) The hyperbola $\dfrac{\left(y+\frac{9}{4}\right)^2}{\left(\frac{3}{4}\right)^2} - \dfrac{x^2}{\left(\frac{3\sqrt{2}}{2}\right)^2} = 1$; (b) the ellipse

$$\frac{\left(x-\frac{3}{2}\right)^2}{\left(\frac{9}{2}\right)^2} + \frac{y^2}{(3\sqrt{2})^2} = 1.$$

37. (a) The circle $x^2 + y^2 = 1$; (b) the straight line that is at right angles to the segment z_1z_2 and passes through its middle; (c) the hyperbola $(x - 1/2)^2 - y^2 = 1/4$; (d) the parabola $y^2 = 2x + 1$.

38. (a) $\bar{z} - z = 0$ and $\bar{z} + z = 0$; (b) $z + \bar{z} + i(z - \bar{z}) = 0$; (c) $k(z + \bar{z}) + 2b + i(z - \bar{z}) = 0$.

39. (a) $z^2 + \bar{z}^2 = 2a^2$; (b) $z\bar{z} + z + \bar{z} = 0$.

40. $-(1+\sqrt[3]{2})$; $-1 + \dfrac{\sqrt[3]{2}}{2} \pm i\dfrac{\sqrt[3]{2}\sqrt{3}}{2}$.

41. -1; 3; $1 \pm 2i$. 42. $z = \dfrac{2z_1 + z_2}{3}$. 43. $b + ai$.

44. $-3 - i\sqrt{3}$, 45. $\sqrt{3} - i$.

46. *Solution.* By hypothesis,

$$(4 - 3i)(\cos\varphi + i\sin\varphi) = \frac{5}{\sqrt{2}}(-1 + i),$$

or

$$\begin{cases} 4\cos\varphi + 3\sin\varphi = -\dfrac{5}{\sqrt{2}}, \\ -3\cos\varphi + 4\sin\varphi = \dfrac{5}{\sqrt{2}}. \end{cases}$$

This means that $\sin\varphi = \sqrt{2}/10$ and $\cos\varphi = -7\sqrt{2}/10$, i.e. $\tan\varphi = -1/7$ and $\varphi = -\arctan(1/7)$.

47. $\varphi = \dfrac{3}{4}\pi$. 48. $x_k = \cot\dfrac{k\pi}{n}$ $(k = 1, 2, \ldots, n-1)$.

49. $x_k = \dfrac{\pi}{4} + k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$.

50. $-\sqrt{2}+i7\sqrt{2}$.

51. $z_1 = 3 + i$, $z_2 = 1 + 3i$, $z_3 = -1 + i$.

53. *Solution.* Consider the sum

$$S_n = (\cos x + i \sin x) + (\cos 2x + i \sin 2x) + \ldots + (\cos nx + i \sin nx).$$

Applying de Moivre's formula, we obtain

$$S_n = (\cos x + i \sin x) + (\cos x + i \sin x)^2 + \ldots + (\cos x + i \sin x)^n.$$

This is the sum of the first n terms of the infinite geometric sequence with a common ratio $q = \cos x + i \sin x$ and a first term $a_1 = \cos x + i \sin x$. This sum is

$$S_n = \frac{(\cos x + i \sin x) - (\cos nx + i \sin nx)(\cos x + i \sin x)}{1 - (\cos x + i \sin x)}$$

Separating the real and imaginary parts, we find that

$$S_n = \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} \cos \frac{(n+1)x}{2} + i \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} \sin \frac{(n+1)x}{2}.$$

Whence

$$\text{(a)}\ \sin x + \sin 2x + \ldots + \sin nx = \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} \sin \frac{(n+1)x}{2};$$

$$\text{(b)}\ \cos x + \cos 2x + \ldots + \cos nx = \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} \cos \frac{(n+1)x}{2}.$$

54. (a) $\frac{\sin^2 nx}{\sin x}$; (b) $\frac{\sin 2nx}{2 \sin x}$.

55. (a) $u = x + 2xy$, $v = y^2 - x^2 - y$; (b) $u = x^2 - y^2$, $v = 1 + 2xy$; (c) $u = 3xy^2 - x^3$, $v = 1 - 3x^2y + y^3$;

(d) $u = \frac{x}{x^2+y^2}$, $v = \frac{y}{x^2+y^2}$; (e) $u = \frac{x - 2xy - y + 1}{(x+1)^2 + y^2}$, $v = \frac{x^2 + y - y^2 + x}{(x+1)^2 + y^2}$; (f) $u = \frac{x^2 - y^2}{x^2 + y^2}$, $v = -\frac{2xy}{x^2+y^2}$.

56. (a) $w = -1$; (b) $w = -3 - 4i$; (c) $w = \frac{1+i}{2}$;

(d) $w = -\frac{5 + 12i}{13}$.

57. (a) The circle $u^2 + v^2 = 4$ traversed clockwise; (b) the v axis (without zero) traversed in the following manner: first from 0 to $+\infty$ and then from $-\infty$ to 0; (c) the ray along the bisector of the third quadrant from ∞ to 0; (d) the ray along the bisector of the first quadrant from ∞ to 0; (e) the bisector of the second quadrant traversed from 0 to ∞ and the bisector of the fourth quadrant traversed from ∞ to 0; (f) the positive half of the real axis traversed from $+\infty$ to 0.

58. (a) The x axis becomes the u axis, and when x varies from $-\infty$ to $+\infty$ the u axis is traversed from $+1$ to $-\infty$ and from $+\infty$ to $+1$ (point 1 is excluded). The y axis becomes a circle, $u^2 + v^2 = 1$. (b) The x axis becomes the u axis, and the latter is traversed in the same manner as in case (a). The y axis becomes a straight line, $u = 1$, traversed from point $+1$ to point $1 + i\infty$ and from point $1 - i\infty$ to point $+1$ (point 1 is excluded).

59. (a) $u = 2x - 1$, $v = 2y$; (b) $u = x^2 - y^2 + x$, $v = (2x + 1)\,y$;

(c) $u = \dfrac{x}{x^2 + y^2}$, $v = -\dfrac{y}{x^2 + y^2}$.

60. (a) $u = e^{-x}\cos y$, $v = -e^{-x}\sin y$; (b) $u = e^{x^2 - y^2}\cos 2xy$, $v = -e^{x^2 - y^2}\sin 2xy$; (c) $u = \sin x \cosh y$, $v = \cos x \sinh y$; (d) $u = \cosh x \cos (y - 1)$, $v = \sinh x \sin (y - 1)$.

61. (a)
$$\begin{cases} u = e^{(x^2 - y^2)\ln 2 - 4k\pi xy}\cos\left[2k\pi (x^2 - y^2) + 2xy \ln 2\right], \\ v = e^{(x^2 - y^2)\ln 2 - 4k\pi xy}\sin\left[2k\pi (x^2 - y^2) + 2xy \ln 2\right] \end{cases}$$
$(k = 0, \pm 1, \pm 2, \ldots)$;

(b) $u = \cosh x \cos y$, $v = \cosh x \sin y$;

(c) $u = \dfrac{\sin x \cos x}{\cosh^2 y - \sin^2 x}$, $v = \dfrac{\sinh y \cosh y}{\cosh^2 y - \sin^2 x}$.

62. (a) $\rho = 3/4$, $\varphi_0 = -\pi/2$; (b) $\rho = 5/4$; $\varphi_0 = \pi$.

63. $\rho = \cosh 1$, $\varphi_0 = \pi/2$. 64. $\rho = \pi$, $\varphi_0 = -\pi/2$.

65. $\rho = \cos^2 (\ln 3)$, $\varphi_0 = 0$.

66. (a) $1 + 2k\pi i$ (here and in what follows $k = 0, \pm 1, \pm 2, \ldots$, if not stated otherwise);

(b) $\left(2k - \dfrac{1}{2}\right)\pi i$; (c) $\left(2k + \dfrac{1}{2}\pi i\right)$; (d) $\ln \sqrt{2} + \left(2k - \dfrac{3}{4}\right)\pi i$;

(e) $\ln \sqrt{13} + \left(2k\pi - \arctan \dfrac{2}{3}\right) i$; (f) $-\left(2k + \dfrac{1}{2}\right)\pi + 2m\pi i$

$(k, m = 0, \pm 1, \pm 2, \ldots)$.

67. (a) $e^{-\left(2k + \frac{1}{2}\right)\pi}$; (b) $e^{\left(2k + \frac{1}{2}\right)\pi}$; (c) $e^{-2k\pi}$;

(d) $e^{\sqrt{2}(2k+1)\pi i}$; (e) $e^{-\left(4k + \frac{1}{2}\right)\pi}$; (f) $e^{(i-1)\left(2k + \frac{1}{6}\right)\pi}$;

(g) $2^{3/2} e^{3\left(2k\pi - \frac{\pi}{4}\right) - 3\left(\frac{\pi}{4} + \ln \sqrt{2} - 2k\pi\right)i}$

68. (a) $\rho = 0$, φ is arbitrary; (b) $\rho = e^{-2k\pi}$, $\varphi = \ln 10 + 2m\pi$ (k, $m = 0, \pm 1, \pm 2, \ldots$,); (c) $\rho = 9e^{2k\pi}$, $\varphi = -\ln 3 + 2m\pi$ (k, $m = 0, \pm 1, \pm 2, \ldots$).

69. (a) $\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$; (b) $\frac{1}{2}\ln 2 - \frac{3}{4}\pi i$.

70. (a) $i \sinh \pi$; (b) $\cosh \pi$; (c) $i \tanh (\pi/2)$.

71. (a) $-i \coth \pi$; (b) $2k\pi - i \ln (\sqrt{2}-1)$, $(2k+1)\pi - i \ln (\sqrt{2}+1)$; (c) $\left(2k+\frac{1}{2}\right)\pi - i \ln (\sqrt{2}+1)$, $\left(2k-\frac{1}{2}\right)\pi - i \ln (\sqrt{2}-1)$.

72. (a) $k\pi + i\frac{\ln 2}{2}$ ($k = 0, \pm 1, \pm 2, \ldots$); (b) i; (c) 0.

73. $z_k = (2k+1)\pi i$ ($k = 0, \pm 1, \pm 2, \ldots$).

74. $z_k = \left(2k-\frac{1}{2}\right)\pi i$ ($k = 0, \pm 1, \pm 2, \ldots$).

75. $z_k = (2k+1)\pi \pm i \ln 2$ ($k = 0, \pm 1, \pm 2, \ldots$).

76. $z_k = \left(k-\frac{1}{2}\right)\pi$ ($k = 0, \pm 1, \pm 2, \ldots$).

77. $z_{2k} = 2k\pi - i \ln (\sqrt{\pi^2+1} - \pi)$, $z_{2k+1} = (2k+1)\pi - i \ln (\sqrt{\pi^2+1} + \pi)$ ($k = 0, \pm 1, \pm 2, \ldots$).

78. $x = 0$.

79. $z_{2k} = 2k\pi i$, $z_{2k+1} = (2k+1)\pi i + \ln 3$ ($k = 0, \pm 1, \pm 2, \ldots$).

80. $z_k = \ln (1+\sqrt{2}) + \left(2k+\frac{1}{2}\right)\pi i$, $z_k = \ln (\sqrt{2}-1) + \left(2k-\frac{1}{2}\right)\pi i$ ($k = 0, \pm 1, \pm 2, \ldots$).

81. (a) $z = 1 - i$; (b) $z = -e + i$. 82. 1. 83. 0. 84. Does not exist. 85. 0. 86. 1/3. 87. $-i$. 88. i. 89. Does not exist. 90. 0. 94. i. 95. $\sqrt{2}$. 96. $-i$. 97. $-2i$.

104. (a) No points of analyticity; (b) analytic; (c) no points of analyticity; (d) analytic; (e) no points of analyticity; (f) analytic.

105. (a) No points of analyticity; (b) no points of analyticity; (c) no points of analyticity; (d) analytic.

109. *Hint.* For any two directions characterized by unit vectors n^0 and s^0 and such that $n^0 = is^0$ the following generalized Cauchy-Riemann conditions are valid:

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial n}, \quad \frac{\partial v}{\partial s} = -\frac{\partial u}{\partial n}. \tag{1}$$

To find the Cauchy-Riemann conditions in polar coordinates,

$$\frac{\partial u}{\partial \rho} = \frac{1}{\rho}\frac{\partial v}{\partial \varphi}, \qquad \frac{\partial v}{\partial \rho} = -\frac{1}{\rho}\frac{\partial u}{\partial \varphi}, \tag{2}$$

we must take the unit vector of the tangent to the circle $|z| = \rho$ directed counterclockwise as s^0 and the unit vector normal to the circle and directed inward as n^0. We must also bear in mind that

$$\frac{\partial}{\partial n} = -\frac{\partial}{\partial \rho}, \qquad \frac{\partial}{\partial s} = \frac{\partial}{\rho\, \partial \varphi}.$$

Then from (1) we can easily find (2). Note that the Cauchy-Riemann conditions in Cartesian coordinates,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

can be found if in (1) we put

$$s^0 = 1, \quad n^0 = i.$$

114. (a) $f(z) = \frac{1}{z}$; (b) $f(z) = \ln z$; (c) $f(z) = z^2 + 2z$.

115. (a) $f(z) = 2 \sinh z - z^2$; (b) $f(z) = 2 \sin z - z$; (c) $f(z) = 4 \cosh z + z^2 - 1$.

116. (a) $f(z) = 2 \cos 2z + z$; (b) $f(z) = 2i(\cos z - 1) - iz^2 + 2$.

118. (a) Yes; (b) no; (c) yes; (d) no.

119. $a + c = 0$.

120. (a) Not a pair of conjugate harmonic functions; (b) a pair of conjugate harmonic functions; (c) not a pair of conjugate harmonic functions; (d) a pair of conjugate harmonic functions.

122. $f(u) = c_1 u + c_2$, with c_1 and c_2 constants.

123. (a) Not harmonic; (b) harmonic; (c) harmonic. (*Hint.* Show that the function $\ln w$ is analytic in D.)

125. (a) A pair of conjugate harmonic functions; (b) not a pair of conjugate harmonic functions; (c) not a pair of conjugate harmonic functions.

126. $u = c_1(ax + by) + c_2$.

127. $u = c_1 xy + c_2$. 128. $u = c_1 \arctan(y/x) + c_2$.

129. $u = c_1(x^2 - y^2) + c_2$.

130. $u = c_1 \sqrt{x + \sqrt{x^2 + y^2}} + c_2$, with c_1 and c_2 arbitrary constants.

131. $u = c_1 \frac{x}{x^2 + y^2} + c_2$.

132. (a) $r_1 = 2$, $\varphi_1 = \frac{\pi}{4}$; $r_2 = \frac{1}{e}$; $\varphi_2 = -\frac{\pi}{2}$; (b) $r_1 = 1$, $\varphi_1 = 0$; $r_2 = \sqrt{\cosh^2 1 - \sin^2 1}$, $\varphi_2 = -\arctan(\tan 1 \tanh 1)$; (c) $r_1 = 15$, $\varphi_1 = -\arctan \frac{4}{3}$; $r_2 = 3\left(1 + \frac{\pi^2}{4}\right)$, $\varphi_2 = \pi - \arctan \frac{4\pi}{\pi^2 - 4}$.

133. The half-plane $\operatorname{Re} z > 0$ stretches and half-plane $\operatorname{Re} z < 0$ shrinks; (b) stretching takes place at all points z inside the circle $|z| = 1$ (except at point $z = 0$) and shrinking takes place at all

points outside this circle; (c) the answer is the same as in (b); (d) the part of the complex plane lying inside the circle $|z| = 1/\sqrt{3}$ shrinks, while that outside the circle stretches.

134. $S_{\tilde{D}} = 8/3$, $l_w = 2(1 + \sqrt{2}) + \ln(3 + 2\sqrt{2})$.

135. $S_{\tilde{D}} = \frac{x_2 - x_1}{4}(\sinh 2y_2 - \sinh 2y_1) - \frac{y_2 - y_1}{4}(\sin 2x_2 - \sin 2x_1)$.

136. 7.5π. 137. $\sqrt{2}(e^{2\pi} - 1)$.

138. The rectangle is mapped into the annulus $e \leqslant |w| \leqslant e^2$, whose area is $\pi(e^4 - e^2)$. But formula (4.9) yields $4(e^4 - e^2)$. The error is due to the fact that under the stated conditions the mapping is not one-to-one.

139. 1/2. 140. $-\frac{\pi}{2}$. 141. $\frac{1}{4}(e^2 - 1)(1 + i)$.

142. (a) $2\pi i$; (b) $-2\pi i$. 143. 0. 144. 0.

145. $(i - 1)e^i$. 146. (a) $2 + i$; (b) $6 + 2i$.

147. $-2(1 + i)$. 148. -1. 149. $\frac{3}{5}(i - 1)$.

150. (a) $e\cos 1 - 1 + ie\sin 1$; (b) $e\cos 1 - 1 + ie\sin 1$.

151. $-(1 + i\sinh 1)$.

152. (a) $2(i - 1)$; (b) $2\sqrt{2}i$.

153. $2\sqrt{2} - 4 + i2\sqrt{2}$. 154. $-7e^{-2} + (3 - 2i)e^i$.

155. $e^{-1} - 1$. 156. $\cos 1 - \sin 1 - ie^{-1}$.

157. $1 - \cos 1 + i(\sin 1 - 1)$.

158. $-\frac{1}{8}\left(\frac{\pi^2}{4} + 3\ln^2 2\right) + i\frac{\pi}{8}\ln 2$.

159. $-\frac{\pi^2}{8}$. 160. $\frac{1}{4}[1 - \cos(2 + 2i)]$.

161. $-\left(\tan 1 + \frac{1}{2}\tan^2 1 + \frac{1}{2}\tanh^2 1\right) + i\tanh 1$.

162. $\sqrt{2\sinh 1} + i(\sqrt{2\sinh 1} - 2\sqrt{\sin 1})$.

163. $\left(\frac{1}{4}\sinh 2 + \frac{1}{2}\right)i$. 164. $-\frac{4}{3}$. 165. 0.

166. $-\ln\sqrt{\sinh^2 1 + \cos^2 1} + i\arctan(\tan 1\tanh 1)$.

167. πi. 168. πe^{-1}. 169. $\frac{\pi}{2}i$. 170. $\pi\sinh 1$.

171. 0. 172. $\frac{2}{3}\pi i\cosh\pi$. 173. 0. 174. $-\frac{\pi}{45}i$.

175. π. 176. 0. 177. $-\pi i$. 178. $2\pi i$.

179. $-\frac{\pi(\pi + 2)\sqrt{2}}{8}i$. 180. 0. 181. $-\frac{\pi i}{27}$. 182. $-\frac{\pi^2}{2}\sinh 1$.

183. $\pi^3 i$. 184. 0. 185. $-2\pi i$. 186. $\frac{1 + i}{2}e^i$.

187. Diverges. 188. Converges. 189. Converges. 190. Converges. 191. Diverges. 192. Is absolutely convergent. 193. Converges. 194. Diverges. 195. Converges. 196. Diverges.

197. $R = 1$. 198. $R = 1$. 199. $R = \sqrt{2}$.

200. $R = \infty$. 201. $R = 1$. 202. $R = \infty$.

203. $R = 1$. 204. $R = 1$. 205. $R = 1$. 206. $R = \infty$.

207. $R = 1$. 208. e^{-1}.

209. (a) and (b): $R \geqslant \dfrac{r + r' - |r - r'|}{2}$; (c) $R \geqslant rr'$;

(d) $R \leqslant \dfrac{r}{r'}$.

210. $-\sin 1 + 2(z+1)\cos 1 + \dfrac{2^2}{2!}(z+1)^2 \sin 1 - \dfrac{2^3}{3!}(z+1)^3 \times$

$\times \cos 1 - \ldots$, $R = \infty$.

211. $\dfrac{1}{\sqrt{2}}\left[1 + \left(z + \dfrac{\pi}{4}\right) - \dfrac{1}{2!}\left(z + \dfrac{\pi}{4}\right)^2 - \dfrac{1}{3!}\left(z + \dfrac{\pi}{4}\right)^3 + \ldots\right]$,

$R = \infty$.

212. $\sqrt{e}\left[1 + \dfrac{1}{2}(2z-1) + \dfrac{1}{2!2^2}(2z-1)^2 + \dfrac{1}{3!2^3}(2z-1)^3 + \ldots\right]$,

$R = \infty$.

213. $-\dfrac{1}{5}\left[1 + \dfrac{3}{5}(z+2) + \dfrac{3^2}{5^2}(z+2)^2 + \dfrac{3^3}{5^3}(z+2)^3 + \ldots\right]$,

$R = \dfrac{5}{3}$.

214. $-\dfrac{1}{5} - \dfrac{9}{25}z - \dfrac{41}{125}z^2 - \ldots$, $R = 1$.

215. $-iz + z^3 + iz^5 - z^7 - \ldots$, $R = 1$,

216. $1 + \dfrac{1}{2}\left(\dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \ldots\right)$, $R = \infty$.

217. $\dfrac{1}{2}\left(\dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \dfrac{z^6}{6!} + \ldots\right)$, $R = \infty$.

218. $\ln 2 - \dfrac{1}{2}\left(\dfrac{z}{2} + \dfrac{z^2}{2 \times 4} + \dfrac{z^3}{3 \times 8} + \ldots\right)$, $R = 2$.

219. $\ln 2 + \dfrac{z}{2} - \dfrac{5z^2}{2 \times 4} + \dfrac{7z^3}{3 \times 8} - \ldots$, $R = 1$.

220. $\dfrac{1}{2} - \dfrac{1}{2^2}z + \dfrac{1}{3!2^3}z^3 + \dfrac{3}{5!2^5}z^5 + \ldots$, $R = \pi$.

221. $\dfrac{1}{2} - \dfrac{1}{2^2}z + \dfrac{1}{2!2^2}z^2 - \dfrac{1}{3!2^3}z^3 + \ldots$, $R = \sqrt{\ln^2(2 - \sqrt{3}) + \pi^2}$.

222. $\dfrac{1}{6} + \dfrac{1}{6^2}z - \dfrac{1}{2!6^3}z^2 + \dfrac{1}{3!6^3}z^3 + \ldots$, $R = \sqrt{\ln^2 5 + \pi^2}$.

223. $\ln 2-\frac{1}{2}z+\frac{1}{2!2^2}z^2-\frac{1}{4!2^3}z^4+\ldots,\quad R=\pi.$

224. $-\frac{1}{2!}z^2-\frac{2}{4!}z^4-\frac{16}{6!}z^6+\ldots,\quad R=\frac{\pi}{2}.$

225. $\ln 2-\frac{z^2}{4}-\frac{z^4}{4\times 4!}-\frac{1}{2\times 6!}z^6+\ldots,\quad R=\pi.$

226. $e\left(1+z+\frac{3}{2!}z^2+\frac{13}{3!}z^3+\ldots\right),\ R=1.$

227. $f(z)=\frac{1}{a-z},\ |z|<1.$ 229. $|z|>1/\sqrt{2}.$

230. $|z|>2.$ 231. $|z|>e^{-1}.$ 232. $|z|>e.$
233. $|z+1|>1/4.$ 234. $|z-2-i|>1/2.$
235. $|z+2i|>3.$ 236. $|z+1-i|>1.$
237. $|z+1+i|<1.$ 238. $|z-i|<2.$
239. $0<|z-2+i|<1.$
240. $2<|z|<4.$ 241. Is divergent everywhere.
242. $1<|z|<2.$
243. $|z-i|>e.$ 244. $1<|z|<2.$
245. $|z+1|>2.$ 246. $0<|z-i|<2.$
247. $0<|z|<1.$ 248. $0<|z-1|<1.$
249. If $|a|>|b|$, the series is divergent everywhere; if $|a|<|b|$, the series converges in the annulus $|a|<|z|<|b|$.

250. $\frac{1}{z}-\frac{z}{3!}+\frac{z^3}{5!}-\frac{z^5}{7!}+\ldots.$

251. $\frac{2}{2!}z-\frac{8}{4!}z^3+\frac{32}{6!}z^5-\ldots.$

252. $\frac{1}{z}+1+\frac{z}{2!}+\frac{z^2}{3!}+\ldots.$

253. $\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{2!z}+\frac{1}{3!}+\frac{z}{4!}+\ldots.$

254. $z^3+z^2+\frac{z}{2!}+\frac{1}{3!}+\frac{1}{4!z}+\ldots.$

255. $z^4-\frac{z^2}{2!}+\frac{1}{4!}-\frac{1}{6!z^2}+\ldots.$

256. $\frac{4^3}{2!2z^3}-\frac{4^4}{4!2z^5}+\frac{4^6}{6!2z^7}-\ldots.$

257. $\frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\frac{z^6}{8!}+\ldots.$

258. $1+\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\ldots.$

259. $\frac{2}{z^4}-\frac{1}{2!z^2}+\frac{1}{4!}-\frac{z^2}{6!}+\ldots.$

260. $\frac{1}{z^2}-\frac{1}{2!z}+\frac{1}{3!}-\frac{z}{4!}+\cdots$.

261. $\frac{1}{z+1}-\frac{1}{(z+1)^2}$.

262. $\frac{\sin 2}{z-2}-\frac{\sin 2}{2!}(z-2)-\frac{\cos 2}{3!}(z-2)^2$

$+\frac{\sin 2}{4!}(z-2)^3+\frac{\cos 2}{5!}(z-2)^4-\cdots$.

263. $(1-i)+(z+i)+\left(\frac{1}{2!}-\frac{i}{1!}\right)\frac{1}{z+i}+\left(\frac{1}{3!}-\frac{i}{2!}\right)\frac{1}{(z+i)^2}+\cdots$

$=\sum_{n=1}^{\infty}\left[\frac{1}{(n+1)!}-\frac{i}{n!}\right](z+i)^{-n}$.

264. (a) $-\sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n}-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$; (b) $\sum_{n=1}^{\infty}\frac{3^{n-1}-2^{n-1}}{z^n}$.

265. (a) $\frac{1}{z}-\sum_{n=0}^{\infty}(-1)^n z^n$; (b) $\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+2}}$.

266. (a) Cannot be expanded;

(b) $\frac{1}{5}\left(\frac{2^3+1}{z^3}-\frac{2^3+2}{z^4}+\frac{2^4-1}{z^5}-\frac{2^5-2}{z^6}+\cdots\right)$.

267. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{z^n}+\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}z^n$

268. (a) $\sum_{n=1}^{\infty}\left[n-\frac{(-1)^n}{2^n}\right]z^{n-1}$;

(b) $\sum_{n=1}^{\infty}\frac{n}{z^{n+1}}+\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}z^n$; (c) $\frac{1}{z}+\sum_{n=1}^{\infty}\frac{n+(-2)^n}{z^{n+1}}$.

269. $\sum_{n=1}^{\infty}\frac{1}{(z+2)^n}-\sum_{n=0}^{\infty}\frac{(z+2)^n}{3^{n+1}}$.

270. Cannot be expanded. 271. $\frac{1}{z-1}+5\sum_{n=2}^{\infty}\frac{2^{n-2}}{(z-1)^n}$.

272. $z+\sum_{n=0}^{\infty}\frac{(n+2)\cdot 4^{n+1}}{z^{2n+1}}$.

273. $-\frac{1}{1}\sum_{n=0}^{\infty}\frac{z^{2n+1}}{4^n}-\frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{z^{2n-1}}$.

274. $-\frac{1}{2(z-i)}+\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2i)^n}(z-i)^n$.

275. $\sum_{n=1}^{\infty}\frac{n\times 4^{n-1}}{(z+2)^{n+3}}$.

276. (a) $z=0$, a second-order zero, and $z_{1,2}=\pm 2i$, simple zeros; (b) $z_n=n\pi$ ($n=\pm 1, \pm 2, \ldots$), simple zeros.

277. (a) $z=0$, a third-order zero, and $z_n=n\pi$ ($n=\pm 1, \pm 2, \ldots$), simple zeros; (b) $z=0$, a simple zero, and $z_n=n\pi i$ ($n=\pm 1, \pm 2, \ldots$ $\ldots$), second-order zeros.

278. (a) $z_n=(2n+1)\pi i$ ($n=0, \pm 1, \pm 2, \ldots$), second-order zeros; (b) $z_n=(4n+1)\pi i/2$ ($n=0, \pm 1, \pm 2, \ldots$), second-order zeros.

279. (a) $z=-\pi i$, a second-order zero, and $z_n=n\pi i$ ($n=0, \pm 1, \pm 2, \ldots$), simple zeros;

(b) $z_n=\sqrt[3]{(2n+1)\frac{\pi}{2}}$ ($n=0, \pm 1, \pm 2, \ldots$) and

$z_n=\sqrt{(2n+1)\frac{\pi}{2}}\,\frac{1\pm i\sqrt{3}}{2}$, simple zeros.

280. (a) $z_{1,2}=\pm\pi i$, second-order zeros, and $z_n=(2n+1)\pi i$ ($n=\pm 1, \pm 2, \pm 3, \ldots$), simple zeros; (b) no zeros.

281. A second-order zero. 282. A third-order zero.

283. A simple zero. 284. A fourth-order zero.

285. A first-order zero. 286. A second-order zero.

287. A fourth-order zero. 288. A fifteenth-order zero.

289. (a) A zero whose order is no less than min (n, m); (b) a zero of order $n+m$; (c) a zero of order $n-m$ if $n>m$, a regular point (not a zero) if $n=m$, or a singular point if $n<m$.

290. (a) A third-order pole; (b) a fourth-order pole; (c) a second order pole.

291. (a) A simple pole; (b) a second-order pole.

292. (a) $z=(4n+1)\frac{\pi}{2}$ ($n=0, \pm 1, \pm 2, \ldots$), second-order poles; (b) $z=0$, a removable singularity.

293. (a) $z=-2$, an essential singularity; (b) $z=0$, an essential singularity.

294. (a) $z = 0$, a second-order pole, and $z = -1$, a second-order pole; (b) $z = 0$, a second-order pole, and $z = 2n\pi i$ ($n = \pm 1, \pm 2, \ldots$), simple poles.

295. (a) $z = 0$, an essential singularity; (b) $z = -1$, an essential singularity; (c) $z = 0$, an essential singularity.

296. (a) $z = 0$, a removable singularity, and $z = 2\pi k$ ($k = \pm 1, \pm 2, \ldots$), second-order poles; (b) $z = \pi/2 + 2k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$), removable singularities, and $z = -\pi/2 + 2k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$), simple poles; (c) $z = \pi$, a simple pole, and $z = k\pi$ ($k = 0, \pm 1, \pm 2, \pm 3, \ldots$), second-order poles.

297. A removable singularity. 298. A simple pole.

299. A simple pole. 300. A removable singularity.

301. An essential singularity.

302. $z = 0$, a fourth-order pole, and $z = -1$, a simple pole.

303. A removable singularity. 304. A removable singularity.

305. A simple pole. 306. A removable singularity.

307. A essential singularity. 308. 1. 309. $-16/3$.

310. 1. 311. -1. 312. 0. 313. 0.

314. $\operatorname{Res} f(0) = 0$, $\operatorname{Res} f\left(\frac{\pi}{4}\right) = \frac{4}{\pi}$,

$$\operatorname{Res} f\left(\frac{\pi}{2} + \pi n\right) = \frac{-8}{\pi^2 (2n+1)(4n+1)} \quad (n = 0, \pm 1, \pm 2, \ldots).$$

315. $\operatorname{Res}(0) = 1/24$.

316. $\operatorname{Res} f(-i) = -\frac{1+3i}{20}\cos 1$, $\operatorname{Res} f(i) = -\frac{1-3i}{20}\cos 1$,

$\operatorname{Res} f(3) = \frac{\cosh 3}{10}$.

317. $$\operatorname{Res} f\left[(-1)^n \frac{\pi}{6} + \pi n\right] = \begin{cases} -\frac{2}{\sqrt{3}} e^{\pi/6 + 2n\pi}, \\ \frac{2}{\sqrt{3}} e^{-\pi/6 - (2n-1)\pi}, \end{cases}$$

$$\operatorname{Res} f\left[(-1)^{n+1} \frac{\pi}{6} + \pi n\right] = \begin{cases} \frac{2}{\sqrt{3}} e^{-\pi/6 + 2n\pi}, \\ -\frac{2}{\sqrt{3}} e^{\pi/6 + (2n-1)\pi} \end{cases}$$

$(n = 0, \pm 1, \pm 2, \ldots)$

318. $\operatorname{Res} f(0) = -\frac{5}{2}$, $\operatorname{Res} f(1) = e$.

319. $\operatorname{Res} f(-1) = \frac{1}{27}$, $\operatorname{Res} f(2) = -\frac{1}{27}$.

320. $\operatorname{Res} f(0) = 0$, $\operatorname{Res} f(z_1) = -\frac{1+i}{4\sqrt{2}} e^i$,

Res $f(z_2)=\dfrac{(1-i)\,e^{-1}}{4\sqrt{2}}$, Res $f(z_3)=\dfrac{(1+i)\,e^{i}}{4\sqrt{2}}$, Res $f(z_4)=$

$=-\dfrac{(1-i)\,e^{-1}}{4\sqrt{2}}$, where z_k $(k=1, 2, 3, 4)$ are the roots of the equation $z^4+1=0$.

321. Res $f(0)=-\dfrac{1}{6}$. 322. Res $f(0)=0$.

323. Res $f(-i)=\dfrac{4}{9}\,i\sinh 2$, Res $f\left(\dfrac{i}{2}\right)=-\dfrac{4}{9}\,(e+2e^{-1})\,i$.

324. Res $f(0)=-\dfrac{1}{6}$, Res $f(3)=\dfrac{2}{27}\sin^3\left(\dfrac{3}{2}\right)$.

325. Res $f(0)=0$.

326. Res $f(-3)=\dfrac{1}{8}\,e^{-3i}$, Res $f(-1)=-\dfrac{e^{-i}}{4}$, Res $f(1)=\dfrac{e^{i}}{8}$.

327. Res $f(0)=-\dfrac{4}{\pi^2}$, Res $f\left(\dfrac{\pi}{2}\right)=0$.

328. Res $f(i)=-1$.

329. Res $f(1)=\dfrac{2n\,(2n-1)\,(2n-2)\,\ldots\,[2n-(n-2)]}{(n-1)!}$.

330. Res $f(n\pi)=0$ $(n=0, \pm 1, \pm 2, \ldots)$.

331. $-\sum\limits_{n=1}^{\infty}\dfrac{1}{(2n-1)!(2n)!}$ at point $z=0$.

332. e at point $z=1$.

333. $\sin 1$ at point $z=0$, and $-\sin 1$ at point $z=1$.

334. $1-e^{-1}$ at point $z=0$, and e^{-1} at point $z=-1$.

335. $e^{-1}-1$ at point $z=0$.

336. $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{(2n)!\,(2n+1)!}$ at point $z=0$.

337. 0. 338. 0. 339. $(1-2e^{-1})\,\pi i$.

340. $2(1-e^{-1})\,\pi i$. 341. $-\dfrac{1}{3}\,\pi i$. 342. 0.

343. $-\dfrac{4}{3}\,\pi i\ln 3$. 344. $2\pi i$.

345. $[\cos 1+\sin 1+i\,(\sin 1-\cos 1)]\,\dfrac{\pi}{2}$.

346. πi. 347. 0. 348. $2\pi i\dfrac{e^2}{3}$. 349. $-\pi^2 i$.

350. $2\pi i$. 351. $\dfrac{\sin 1-4\cos 1}{12}\,\pi i$.

352. $-\frac{\pi i}{\sqrt{2}}$. 353. 0. 354. $3\pi i$. 355. 0.

356. $z = \infty$, a simple pole.
357. $z = \infty$, a removable singularity.
358. $z = \infty$, an essential singularity.
359. $z = \infty$, a removable singularity.
360. $z = \infty$, a removable singularity.
361. $z = \infty$, a third-order pole.
363. $2\pi i$. 364. 0. 365. 0. 366. $2\pi ei$.

367. $-\frac{\pi}{3} i$. 368. $2\pi i$. 369. $\frac{\pi}{\sqrt{2}}$. 370. $\frac{\pi}{ab(a+b)}$.

371. $\frac{3}{8}\pi$. 372. $\frac{(2n)!}{(n!)^2} 2^{-2n}\pi$. 373. $-\frac{\pi}{27}$.

374. $\frac{\pi}{2(b^2-a^2)^3}\left(\frac{5b^2-a^2}{b^3}+\frac{b^2-5a^2}{a^3}\right)$.

375. $\frac{2}{3}\pi$. 376. $\frac{\pi}{n \sin \frac{2m+1}{n}\pi}$. 377. $\frac{2}{3}\pi$.

378. $\frac{\pi}{2}$. 379. $\frac{\pi}{16a^{3/2}b^{5/2}}$.

381. $\frac{\pi}{3} e^{-3}(\cos 1 - 3 \sin 1)$.

382. $\frac{\pi}{2} e^{-4}(2 \cos 2 + \sin 2)$.

383. $\frac{\pi}{12} e^{-2}(2e-1)$. 384. $\frac{\pi}{3} e^{-3}$.

385. $\frac{\pi}{2\sqrt{2}} e^{-a/\sqrt{2}}\left(\cos \frac{a}{\sqrt{2}} + \sin \frac{a}{\sqrt{2}}\right)$.

386. $\frac{\pi}{2a} e^{-a}$. 387. $\frac{\pi}{2a} e^{-ma}$.

388. $\frac{\pi}{\sqrt{3}} e^{-\frac{\sqrt{3}}{2}} \sin \frac{1}{2}$. 389. $\frac{\pi}{16}\left(e^{-\lambda} - \frac{1}{3} e^{-3\lambda}\right)$.

390. $\frac{\pi}{4}(2-a) e^{-a}$. 391. 0.

392. $\frac{\pi}{4b^3} e^{-bm}[3b^2 - a^2 - mb(3b^2 + a^2)]$. 393. $\frac{\pi}{2}$.

394. $\frac{\pi}{2}(1 - e^{-1})$. 395. $\frac{b-a}{2}\pi$.

396. $\frac{\pi}{2a^2} - \frac{\pi e^{-ma}}{4a^3}\left(m + \frac{2}{a}\right)$.

397. $\frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-\frac{b^2}{4a}}$.

398. $\pi\left(\frac{1}{\sin a\pi}-\frac{1}{\sin b\pi}\right)$. 399. $\frac{2\pi}{1-p^2}$.

400. $\frac{\pi(1-p+p^2)}{1-p}$. 401. $\frac{2\pi}{p^2(p^2-1)}$. 402. 0.

403. $\frac{2\pi}{\sqrt{a^2-1}}$. 404. πi.

405. $\frac{2\pi}{b^2}(a-\sqrt{a^2-b^2})$. 406. $\frac{2\pi}{\sqrt{1-a^2}}$.

407. $\frac{2\pi}{\sqrt{a^2-b^2}}$. 408. $\frac{1-\pi a\cot\pi a}{2a^2}$.

409. $\frac{1}{2a^4}-\frac{\pi}{4a^3}(\cot\pi a+\coth\pi a)$.

410. $\frac{\pi^2}{\sin^2\pi a}$. 411. $\frac{\pi^2}{8}$. 412. $\frac{\pi^2\cot\pi a}{\sin\pi a}$.

413. $\frac{\pi^3}{32}$. 414. $\frac{\pi\cosh\alpha a}{2a\sinh\pi a}+\frac{1}{2a^2}$.

415. $\frac{1}{4a^4}\left(\frac{\pi^2a^2\cosh\pi a}{\sinh\pi a}-\frac{\pi a}{\sinh\pi a}-2\right)$.

416. $\operatorname{Res}_{z=k\pi}\frac{f'(z)}{f(z)}=1 \quad (k=\pm1,\pm2,\ldots)$.

417. $\operatorname{Res}_{z=\frac{\pi}{2}+k\pi}\frac{f'(z)}{f(z)}=3 \quad (k=0,\pm1,\pm2,\ldots)$.

418. (a) $\operatorname{Res}_{z=0}\frac{f'(z)}{f(z)}=-1$, $\operatorname{Res}_{z=\frac{\pi}{2}+k\pi}\frac{f'(z)}{f(z)}=1 \quad (k=0,\pm1,\pm2,\ldots)$;

(b) $\operatorname{Res}_{z=k\pi}\frac{f'(z)}{f(z)}=1 \quad (k=0,\pm1,\pm2,\ldots)$.

419. -2. 420. 3. 421. 6. 422. -1. 423. -3. 424. -4. 425. 2. 426. 1. 427. 1. 428. 1.

429. 2. 430. 6.

431. 3. 432. No roots. 433. 5. 434. No roots.

435. 11. 436. 6. 437. 2. 438. 3. 439. 4.

440. 1. 441. n. 442. 2. 443. 4. 444. No roots.

445. 1. 446. 1.

450. (a) The entire plane; (b) the entire plane without point $z=2$; (c) the entire plane without point $z=0$; (d) the entire plane without points $z_k=1-(k+1/2)\pi i$, $k=0,\pm1,\pm2,\ldots$; (e) the entire plane without point $z=-2i$.

454. (a) and (b), translation; (c), (d), and (f), rotation; (e) stretching.

455. (a) $w = az + b$; (b) $w = -az + b$; (c) $w = -i(az + b)$, where a and b are real; $a > 0$.

456. (a) $w = -\frac{1+8i}{5} z + \frac{14+2i}{5}$; (b) $w = 2z + i$; (c) $w = iz - 2$.

457. $w = \frac{z-a}{h}$.

458. *Hint.* Putting $z = x + iy$ and $w = u + iv$, we obtain

$$u = \frac{x}{x^2+y^2}, \quad v = -\frac{y}{x^2+y^2}.$$

Traversing the boundary of the half-strip in such a way that, say, the domain remains to the left, we find, by the principle of one-to-one correspondence between boundaries, that the half-strip is mapped into the first quadrant without the semicircle $|z - 1/2| \leqslant 1/2$.

459. (a) $\arg w = -\frac{\pi}{3}$; (b) $|w| = 1$, $-\pi < \arg w < -\frac{\pi}{4}$; (c) $\frac{1}{4} \leqslant u \leqslant \frac{1}{2}$, $v = 0$; (d) $\frac{1}{2} < v < 1$; $u = 0$; (e) $\left|w - \frac{1}{2}\right| > \frac{1}{2}$, $u > 0$.

460. $w = \frac{z-i}{iz-i}$. 461. $\left|w - \frac{1}{3}\right| > \frac{1}{3}$, $u < \frac{3}{4}$.

462. $\operatorname{Re} w > 0$. 463. $u < v$. 464. $u + v < 0$.

465. (a) $-1 + i$; (b) $1 + i$; (c) ∞.

466. (a) $w = -az + b$; (b) $w = -i(az + b)$, where a and b are real, $a > 0$.

467. $w = \frac{2}{2-z}$. 468. (a) $w = \frac{z-i}{z+i}$; (b) $w = i\frac{z-2i}{z+2i}$.

469. $w = \frac{z-i}{iz-1}$. 470. $w = \frac{i-1-(1+i)z}{(3i-1)z-1-3i}$.

471. $w = i\frac{1-z}{1+z}$. Use formula (12.7).

472. $w = \frac{2z-5}{10-z}$. Use formula (12.7).

473. *Solution.* Use formula $w = e^{i\varphi}(z - z_0)/(1 - \bar{z}z_0)$, where z_0 is the point of the first circle mapped into the center of the second. In the case (a) we have $f(1/2) = 0$, i.e. $z_0 = 1/2$. Using the condition that $\arg f'(1/2) = \pi/2$, we obtain $\varphi = \pi/2$. In the case (b) we find, similarly, that $w = -iz$.

474. $w = \frac{5-z+\sqrt{5}(i-1)(z+1)}{5-z+\sqrt{5}(i+1)(z+1)}$.

475. The first quadrant of the complex w plane.

476. The domain $1/2 \leqslant |w-1| \leqslant 1$, $-\pi/4 \leqslant \arg(w-1) \leqslant 0$.

477. $w = \sqrt{\dfrac{1+z}{1-z}}$. 478. $w = \sqrt{e^{2\pi z i} + e^{-2\pi a}}$.

479. $w = \sqrt{\dfrac{z-b}{z-a}}$. 480. $w = \left(\dfrac{z-i}{iz-1}\right)^2$.

Hint. First map the circle into the upper half-plane and then map the latter into the plane with the cut.

481. $w = [(z-z_0)\, e^{-i\varphi_1}]^{\frac{\pi}{\varphi_2-\varphi_1}}$.

482. $w = -e^{-z}$. 483. $w = \left(\dfrac{e^{-z}-1}{e^{-z}+1}\right)^2$.

484. Into the rectangle $\{\ln r \leqslant u \leqslant \ln R,\ 0 \leqslant v \leqslant \pi\}$.

485. $w_1 = ze^{-i\frac{\pi}{2}}$, $w = \sqrt{w_1} = \sqrt{z}e^{-i\frac{\pi}{4}}$ $(\sqrt{1} = 1)$.

486. $w_1 = \dfrac{z-a}{b-z}$, $w = \sqrt{w_1} = \sqrt{\dfrac{z-a}{b-z}}$.

487. $w_1 = z-a$, $w_2 = \dfrac{w_1}{w_1-(b-a)}$, $w = \sqrt{w_2} = \sqrt{\dfrac{z-a}{z-b}}$.

488. $w_1 = z^2$, which means that $w_1|_{z=1+i} = 2i$. According to (12.9), $w = e^{i\varphi}\dfrac{w_1-2i}{w_1+2i} = e^{i\varphi}\dfrac{z^2-2i}{z^2+2i}$.

From the fact that $w(0) = 1$ we find that $e^{i\varphi} = -1$. The final result is $w = (2i - z^2)/(2i + z^2)$.

489. The upper semicircle $|w-1| < 1$, $\operatorname{Im} w > 0$.

490. A quarter of a circle, $|w| < 1$, $0 < \arg w < \pi/2$.

491. A rectangle with vertices at points 1, 2, $2 + ie$, and $1 + ie$.

492. $w_1 = z^4$, $w_2 = \dfrac{16+w_1}{16-w_1}$, $w = w_2^2 = \left(\dfrac{16+z^4}{16-z^4}\right)^2$.

493. $w_1 = z-a$, $w_2 = \dfrac{\pi i}{b-a}w_1$, $w = e^{w_2} = e^{\frac{\pi i(z-a)}{b-a}}$.

494. $w = \ln z$.

495. $w_1 = e^{-z}$, $w_2 = -w_1$, $w_3 = \dfrac{w_2+1}{1-w_2}$, $w = w_3^2 = \left(\dfrac{1-e^{-z}}{1+e^{-z}}\right)^2$.

496. $w_1 = 2iz$, $w_2 = w_1 + \dfrac{\pi}{2}$, $w = \sin w_2 = \sin\left(2iz + \dfrac{\pi}{2}\right) = \cosh 2z$.

497. $w = 2\dfrac{z-2}{z}$. 498. $w = 2i\dfrac{z+2}{z}$.

499. $w = \left(\dfrac{z}{z-1+i}\right)^2$. 500. $w = 2\left(z + \dfrac{1}{z}\right)$.

501. $w_1 = 2z$, $w_2 = \frac{1}{2}\left(w_1 + \frac{1}{w_1}\right)$, $w = \frac{w_2 - i}{w_2 + i} = \frac{4z^2 - 4iz + 1}{4z^2 + 4iz + 1}$.

502. $w = -\frac{z^8 - i}{z^8 + i}$. 503. $w = e^{i2z}$.

504. (a) The unit circle $|w| < 1$; (b) the entire w plane with a cut along the segment $u = 0$, $-1 \leqslant v \leqslant 1$.

505. The velocity potential is $u = x^2 - y^2 + 2x + 2$, the stream function is $v = 2(x + 1)y$, the equipotential curves are $x^2 - y^2 + 2x = c_1$ (hyperbolas), the streamlines are $xy + y = c_2$ (hyperbolas), the magnitude of the velocity is $V = 2\sqrt{(x+1)^2 + y^2}$, the direction of the velocity vector is given by formula (1.1):

$$\varphi = -\arctan\frac{y}{x+1} + m\pi, \quad m = -1, 0, 1,$$

and the velocity projections on the x and y axes are $V_{Ox} = 2(x + 1)$ and $V_{Oy} = -2y$.

506. The velocity potential is $u = \frac{x^2 - y^2}{(x^2+y^2)^2}$, the stream function is $v = -\frac{2xy}{(x^2+y^2)^2}$, the equipotential curves are $x^2 - y^2 = c_1(x^2+y^2)^2$, the streamlines are $xy = c_2(x^2+y^2)^2$, the magnitude of the velocity is $V = \frac{2}{(x^2+y^2)^{3/2}}$, the direction of the velocity vector is given by formula (1.1):

$$\varphi = 3\arctan\frac{y}{x} + (3m - 1)\pi, \quad m = -1, 0, 1,$$

and the velocity projections on the x and y axes are

$$V_{Ox} = \frac{2x(3y^2 - x^2)}{(x^2+y^2)^3}, \quad V_{Oy} = \frac{2y(y^2 - 3x^2)}{(x^2+y^2)^3}.$$

507. The velocity potential is $u = \frac{1}{2}\ln[(x-1)^2 + y^2]$, the stream function is $v = \arctan\frac{y}{x-1}$, the equipotential curves are $(x-1)^2 + y^2 = c_1$ (circles), the streamlines are $y = c_2(x - 1)$ (straight lines), the magnitude of the velocity is $V = \frac{1}{\sqrt{(x-1)^2+y^2}}$, the direction of the velocity vector is given by

$$\varphi = \arctan\frac{y}{x-1} + m\pi, \quad m = -1, 0, 1,$$

and the velocity projections on the x and y axes are

$$V_{Ox} = \frac{x-1}{(x-1)^2+y^2}, \quad V_{Oy} = \frac{y}{(x-1)^2+y^2}.$$

508. $f(z) = (1 - i)z^2 + z$. 509. $f(z) = \sin z + c$.

510. $\Gamma_L = -10\pi$ along both circles.

511. (a) Yes; (b) yes; (c) no; (d) yes; (e) yes; (f) no; (g) no; (h) yes; (i) no; (j) yes; (k) yes; (l) yes.

512. $\frac{1}{p^2}$. 513. $\frac{3}{p^2+9}$. 514. $\frac{1}{(p-1)^2}$.

515. $\frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$. 516. No. 517. $\frac{p+1}{p^2}$.

518. $\frac{2-p}{p^2+1}$. 519. $\frac{p^2+2p+2}{2p^2(p+1)}$. 520. $\frac{1}{p-a}$.

521. $\frac{4}{p^2+16}$. 522. (a) $\frac{p}{p^2+\omega^2}$; (b) $\frac{3}{p^2-9}$.

523. $aF(pa)$. 524. $\frac{2}{p(p^2+4)}$.

525. $\frac{m(p^2+m^2-n^2)}{(p^2+m^2+n^2)^2-4m^2n^2}$. 526. $\frac{p^3+7p}{(p^2+9)(p^2+1)}$.

527. $\frac{2mnp}{(p^2+m^2-n^2)^2-4m^2n^2}$.

528. $\frac{1}{8}\left(\frac{3}{p}+\frac{p}{p^2+16}-\frac{4p}{p^2+1}\right)$.

529. $\frac{p(p^2+m^2+n^2)}{(p^2+m^2+n^2)^2-4m^2n^2}$.

530. $\frac{p^2+2}{p(p^2+4)}$. 531. $\frac{6}{(p^2+1)(p^2+9)}$.

532. $\frac{2\omega p}{(p^2+\omega^2)^2}$. 533. $\frac{p^4+16p^2+24}{p(p^2+4)(p^2+16)}$.

534. $\frac{p^2-\omega^2}{(p^2+\omega^2)^2}$. 535. $\frac{1}{(p-1)^2}$.

536. $\frac{2p^3-6p}{(p^2+1)^3}$. 537. $\frac{2(p^2+p+1)}{(p^2-1)^2}$.

538. $\frac{2p^2+4p+8}{(p^2+4)^2}$. 539. $\frac{6p}{(p^2-1)^2}$.

540. $\frac{1}{p(p^2+1)}$. 541. $\frac{p^3+p^2+p\omega^2-\omega^2}{p(p^2+\omega^2)^2}$.

542. $\frac{4}{(p^2-4)^2}$. 543. $\frac{p^2+2\omega^2}{p^2(p^2+4\omega^2)}$.

544. $\frac{1}{p^2-\omega^2}$. 545. $\frac{2}{p(p+1)^3}$.

546. (a) $\ln\frac{p}{p-1}$; (b) $\ln\frac{p+1}{p}$; (c) $\frac{1}{2}\ln\frac{\sqrt{p^2+4}}{p}$.

547. (a) $\ln \frac{\sqrt{p^2+1}}{p}$; (b) $\frac{1}{2} \ln \frac{p^2+4}{p^2+1}$.

548. (a) $\ln \frac{p}{p-1} - \frac{1}{p}$; (b) $\ln \frac{p+1}{p-1}$.

549. $\ln \frac{b}{a}$. 550. $\arctan \frac{a}{\alpha}$.

551. $\arctan \frac{\beta}{m} - \arctan \frac{\alpha}{m}$.

552. $A \ln \frac{\delta}{\alpha} + B \ln \frac{\delta}{\beta} + C \ln \frac{\delta}{\gamma}$. 553. $\ln \frac{b}{a}$.

554. $\frac{1}{2} \ln \left| \frac{a+b}{a-b} \right|$.

555. (a) $\frac{1}{(p-2)^2-1}$; (b) $\frac{p-m}{(p-m)^2+n^2}$.

556. $\frac{3!}{(p+1)^4}$. 557. $\frac{1}{(p-1)^2-1}$. 558. $\frac{p^2-2p}{(p^2-2p+2)^2}$.

559. $\frac{1}{2(p-3)} - \frac{1}{2} \cdot \frac{p-3}{(p-3)^2+4}$.

560. $\frac{1}{2(p+\alpha)} + \frac{p+\alpha}{2[(p+\alpha)^2+4\beta^2]}$.

561. $\frac{e^{-bp}}{p^2+1}$. 562. $\frac{e^{-bp}}{2p} + \frac{pe^{-bp}}{2(p^2+4)}$.

563. $\frac{e^{-2p}}{p-1}$. 564. $\frac{1-e^{-p}}{p}$. 565. $\frac{1-2e^{-p}+e^{-2p}}{p}$.

566. $\frac{1-2e^{-p}+e^{-2p}}{p^2}$. 567. $\frac{e^{-ap}}{p+b}$. 568. $\frac{be^{-ap}}{p(p+b)}$.

569. $\frac{1-e^{-ap}}{p^2}$. 570. $\frac{e^{-ap}-e^{-bp}}{p^2}$.

571 $\frac{1}{p} \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)}{e^{kp}}$.

572. $F(p) = \frac{1}{ap^2}(2e^{-2ap}-1) + \frac{2}{p} e^{-ap}$

573. $F(p) = \frac{e^{-ap}}{p}(2-e^{-ap}-e^{-2ap})$.

574. $F(p) = -\frac{1}{p} + \frac{1}{ap^2} + \frac{ap-2}{ap^2} e^{-ap} + \frac{1}{ap^2} e^{-2ap}$.

575. $F(p)=\frac{1}{p}+\frac{1-2ap}{ap^2}e^{-ap}-\frac{1}{ap^2}e^{-2ap}$.

576. $F(p)=\frac{1}{p}-\frac{1}{ap^2}e^{-ap}+\frac{2}{ap^2}e^{-3ap}$.

577. $F(p)=-\frac{b}{p}e^{-ap}+\frac{2b}{p}e^{-2ap}$.

579. $F(p)=\frac{(1-e^{-p})^2}{p^2(1-e^{-4p})}$.

580. $\frac{1+p-e^{-p}}{p^2(e^p-1)}$. 581. $\frac{1}{p^2+1}\cdot\frac{1+e^{-\pi p}}{1-e^{-\pi p}}$.

582. $\frac{1}{p^2+1}\left(p+\frac{2e^{-\frac{\pi}{2}p}}{1-e^{-\pi p}}\right)$. 583. $\frac{1}{(p^2+1)(1-e^{-\pi p})}$.

587. (a) $\frac{2e^{-\frac{\pi}{8}p}}{p^2+4}$; (b) $\frac{pe^{-\frac{\pi}{18}p}}{p^2+9}$; (c) $\frac{3e^{-2p}}{p^2-9}$.

588. $\sum_{k=0}^{\infty} m_k e^{-kp}$. 589. $\frac{1}{(p-1)(p^2+1)}$.

590. $\frac{p}{(p-2)(p^2+1)}$. 591. $\frac{2}{p^2(p^2-1)}$.

592. $\frac{n!\,F(p)}{p^{n+1}}$. 593. $\frac{2}{p^3(p+2)}$.

595. *Solution.* Since $J_1(t)=-J_0'(t)$, by employing the results obtained in Problem 594 and the property of differentiation of object function we find that

$$J_1(t)\doteq -p\,\frac{1}{\sqrt{p^2+1}}+J_0(0)=-\frac{p}{\sqrt{p^2+1}}+1=\frac{\sqrt{p^2+1}-p}{\sqrt{p^2+1}}.$$

596. *Solution.* For $n=0$ and $n=1$ the formula

$$J_n(t)\doteq\frac{(\sqrt{p^2+1}-p)^n}{\sqrt{p^2+1}}$$

is valid. Now we will apply the method of mathematical induction. Since

$$2J_n'(t)=J_{n-1}(t)-J_{n+1}(t) \text{ and } J_{n-1}(0)=0 \quad (n\geqslant 2),$$

we arrive at the following result:

$$J_n(t)=J_{n-2}(t)-2J_{n-1}'(t)$$

$$\doteq\frac{(\sqrt{p^2+1}-p)^{n-2}}{\sqrt{p^2+1}}-\frac{2p(\sqrt{p^2+1}-p)^{n-1}}{\sqrt{p^2+1}}=\frac{(\sqrt{p^2+1}-p)^n}{\sqrt{p^2+1}}.$$

597. *Solution.* We consider the function

$$F(p)=\frac{1}{p^{n+1}}e^{-\frac{1}{p}} \quad (n=0, 1, 2, \ldots).$$

We have

$$F(p)=\frac{1}{p^{n+1}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,p^k}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,p^{n+k+1}}.$$

Consequently, $f(t)=\sum_{k=0}^{\infty}\frac{(-1)^k t^{n+k}}{k!\,(n+k)!}$. Noting that

$$J_n(2\sqrt{t})=\sum_{k=0}^{\infty}\frac{(-1)^k t^{\frac{n}{2}+k}}{k!\,(n+k)!},$$

we find that $f(t)=t^{n/2}J_n(2\sqrt{t})$. For $n=0$ we have

$$J_0(2\sqrt{t})\doteqdot e^{-1/p}.$$

599. We put $\varphi(t)=t^n e^{-t}$. By the property of translation of result function,

$$t^n e^{-t}\doteqdot\frac{n!}{(p+1)^{n+1}}.$$

We can easily see that $\varphi(0)=\varphi'(0)=\ldots=\varphi^{(n-1)}(0)=0$. By the property of differentiation of object function,

$$\frac{d^n}{dt^n}(t^n e^{-t})\doteqdot\frac{p^n\times n!}{(p+1)^{n+1}}.$$

Using the property of translation of result function, we find that

$$L_n(t)=\frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t})\doteqdot\frac{(p-1)^n\times n!}{n!\,p^{n+1}}$$

$$=\frac{1}{p}\left(1-\frac{1}{p}\right)^n \quad (n=0, 1, 2, \ldots).$$

600. $-\frac{\ln p}{p}-\frac{C}{p}$, with $C=\lim_{n\to\infty}(1+1/2+\ldots+1/n-\ln n)$ the Euler-Mascheroni constant, approximately equal to 0.577216.

601. *Solution.* We take the function $f(t)=e^t\operatorname{erf}(\sqrt{t})$ and assume that $F(p)$ is the result function for $f(t)$. We have $(\operatorname{erf} t)'=(2/\sqrt{\pi})e^{-t^2}$ and

$$f'(t)=e^t\operatorname{erf}(\sqrt{t})+\frac{1}{\sqrt{\pi t}}=f(t)+\frac{1}{\sqrt{\pi t}}. \quad (1)$$

Going over to result functions and bearing in mind that $f(0) = 0$, we can use (1) to obtain $pF(p) = F(p) + 1/\sqrt{p}$, whence $F(p) = [(p-1)\sqrt{p}]^{-1}$. Here we used the result obtained in Problem 515 and the fact that $\Gamma(1/2) = \sqrt{\pi}$. Thus,

$$e^t \operatorname{erf}(\sqrt{t}) \doteqdot \frac{1}{(p-1)\sqrt{p}}.$$

Applying the property of translation of result function, we find that

$$\operatorname{erf}(\sqrt{t}) \doteqdot \frac{1}{p\sqrt{p+1}}.$$

607. (a) $\frac{\pi}{2} e^{-t}$; (b) $\frac{\pi}{2} \eta(t-1)$.

608. $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$.

609. $(t-1)^2 \eta(t-1)$. 610. $(t-2) \eta(t-2)$.

611. $e^{t-2} \eta(t-2)$. 612. $e^{-3(t-3)} \eta(t-3)$.

613. $e^{-2t} \sin t$. 614. $\frac{1}{2}(e^{-t} - e^{-3t})$.

615. $(1-t) e^{-t}$. 616. $\frac{1}{2} t \sin t$.

617. $1 - e^{-t} - te^{-t}$. 618. $\frac{2\sqrt{3}}{9} e^{\frac{t}{2}} \sin \frac{3\sqrt{3}}{2} t$.

619. $\frac{t^2}{2} + 2e^{-t} \sin t$. 620. $t - \sin t$.

621. $\frac{1}{6} e^{2t} - \frac{1}{15} e^{-t} - \frac{1}{10} \cos 2t - \frac{1}{5} \sin 2t$.

622. $1 - ne^{-t} + \frac{1}{2} n(n-1) e^{-2t} - \ldots + (-1)^n e^{-nt}$.

623. $\frac{2}{3} e^{-t/2} \left[\frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t - t \cos \frac{\sqrt{3}}{2} t \right]$.

624. $e^{-t}(1-t^2)$.

625. $\frac{1}{3} e^{t/2} \left(\cos \frac{\sqrt{3}}{2} t + \sqrt{3} \sin \frac{\sqrt{3}}{2} t \right) - \frac{1}{3} e^{-t}$.

626. $\frac{3}{5} + \frac{e^{-2t}}{5} (4 \sin t - 3 \cos t)$. 627. $\frac{1}{9} (e^{-2t} - e^t + 3te^t)$.

628. $2e^t + e^{t/2} \left(\frac{5\sqrt{3}}{3} \sin \frac{\sqrt{3}}{2} t - \cos \frac{\sqrt{3}}{2} t \right)$.

629. $\frac{1}{3} te^t - \frac{1}{3} te^{-t/2} \left(\cos \frac{\sqrt{3}}{2} t + \sqrt{3} \sin \frac{\sqrt{3}}{2} t \right)$.

630. $\frac{1}{2}e^{t-1}\sin 2(t-1)\,\eta(t-1)+\cos 3(t-2)\,\eta(t-2)$.

631. $(t-3)\,e^{-(t-3)}\eta(t-3)$.
632. $e^{t-1}\eta(t-1)-\eta(t-1)$.
633. $\sin(t-2)\,\eta(t-2)+2\sin(t-3)\,\eta(t-3)+$
$+3\sin(t-4)\,\eta(t-4)$.
634. $\sinh(t-1)\,\eta(t-1)+\cosh 2(t-2)\,\eta(t-2)$.

635. $\frac{1}{4}\eta\left(t-\frac{1}{2}\right)-\frac{1}{5}e^{-\left(t-\frac{1}{2}\right)}\eta\left(t-\frac{1}{2}\right)-\frac{1}{20}\cos 2\left(t-\frac{1}{2}\right)\eta\times$
$\times\left(t-\frac{1}{2}\right)-\frac{1}{10}\sin 2\left(t-\frac{1}{2}\right)\eta\left(t-\frac{1}{2}\right)$.

636. $(t-1)\,\eta(t-1)+(t-2)^2\eta(t-2)+(t-3)^3\,\eta(t-3)$.

637. $\eta\left(t-\frac{1}{3}\right)-\cos\left(t-\frac{1}{3}\right)\eta\left(t-\frac{1}{3}\right)$.

638. $1-\operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right)$.

639. *Solution.* $\frac{e^{-a\sqrt{p}}}{p\sqrt{p}}=\frac{1}{\sqrt{p}}\frac{e^{-a\sqrt{p}}}{(\sqrt{p})^2}$. We assume that $\Phi(p)=\frac{1}{\sqrt{p}}$ and $F(\sqrt{p})=\frac{e^{-a\sqrt{p}}}{(\sqrt{p})^2}$.

Whence, $F(p)=\frac{e^{-ap}}{p^2}$ and, by the property of translation of object function,

$$F(p)\doteq(t-a)\,\eta(t-a)=f(t).$$

By Êfros's theorem,

$$\frac{e^{-a\sqrt{p}}}{p\sqrt{p}}\doteq\frac{1}{\sqrt{\pi t}}\int_0^\infty(\tau-a)\,\eta(t-a)\,e^{-\frac{\tau^2}{4t}}\,d\tau$$

$$=\frac{1}{\sqrt{\pi t}}\int_a^\infty(\tau-a)\,e^{-\frac{\tau^2}{4t}}\,d\tau=\frac{1}{\sqrt{\pi t}}\int_a^\infty\tau e^{-\frac{\tau^2}{4t}}\,d\tau$$

$$-\frac{a}{\sqrt{\pi t}}\int_a^\infty e^{-\frac{\tau^2}{4t}}\,d\tau=I_1(t)+I_2(t).$$

Here

$$I_1(t)=\frac{1}{\sqrt{\pi t}}\int_a^\infty \tau e^{-\frac{\tau^2}{4t}}\,d\tau=\frac{1}{\sqrt{\pi t}}(-2t)\,e^{-\frac{\tau^2}{4t}}\Big|_{\tau=a}^{\infty}$$

$$=2\sqrt{\frac{t}{\pi}}\,e^{-\frac{a^2}{4t}},$$

$$I_2(t)=-\frac{a}{\sqrt{\pi t}}\int_a^\infty e^{-\frac{\tau^2}{4t}}\,d\tau$$

$$=\frac{a}{\sqrt{\pi t}}\int_0^a e^{-\frac{\tau^2}{4t}}\,d\tau-\frac{a}{\sqrt{\pi t}}\int_0^\infty e^{-\frac{\tau^2}{4t}}\,d\tau$$

$$=a\left(\frac{2}{\sqrt{\pi}}\int_0^{\frac{a}{2\sqrt{t}}} e^{-s^2}\,ds-\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-s^2}\,ds\right),$$

where $s=\tau/2\sqrt{t}$. Consequently, $I_2(t)=a\left[\operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right)-1\right]$

$=-a\operatorname{Erf}\left(\frac{a}{2\sqrt{t}}\right)$, and finally

$$\frac{e^{-a\sqrt{p}}}{p\sqrt{p}}\doteqdot 2\sqrt{\frac{t}{\pi}}\,e^{-\frac{a^2}{4t}}-a\operatorname{Erf}\left(\frac{a}{2\sqrt{t}}\right).$$

640. $\left(t+\frac{\alpha^2}{2}\right)\operatorname{Erf}\left(\frac{\alpha}{2\sqrt{t}}\right)-\alpha\sqrt{\frac{t}{\pi}}\,e^{-\frac{\alpha^2}{4t}}$.

641. $ae^{hx+a^2h^2t}\operatorname{Erf}\left(\frac{x}{2a\sqrt{t}}+ah\sqrt{t}\right)$.

642. *Solution.* $\frac{e^{-\alpha\sqrt{p}}}{p(a+\sqrt{p})}=\frac{1}{\sqrt{p}}\,\frac{e^{-\alpha\sqrt{p}}}{\sqrt{p}(a+\sqrt{p})}$. We take

$$\Phi(p)=\frac{1}{\sqrt{p}},\quad F(\sqrt{p})=\frac{e^{-\alpha\sqrt{p}}}{\sqrt{p}(a+\sqrt{p})}.$$

Whence,

$$F(p)=\frac{e^{-\alpha p}}{p(a+p)}=\frac{1}{a}\left(\frac{e^{-\alpha p}}{p}-\frac{e^{-\alpha p}}{p+a}\right)$$

$$\doteqdot\frac{1}{a}\left[\eta(t-\alpha)-e^{-a(t-\alpha)}\eta(t-\alpha)\right]=f(t).$$

By Efros's theorem,

$$\frac{e^{-\alpha\sqrt{p}}}{p(a+\sqrt{p})} \doteqdot \frac{1}{a\sqrt{\pi t}} \int_{\alpha}^{\infty} [1-e^{-a(\tau-\alpha)}]\, e^{-\frac{\tau^2}{4t}}\, d\tau$$

$$= \frac{1}{a}\frac{2}{\sqrt{\pi}} \int_{\frac{\alpha}{2\sqrt{t}}}^{\infty} e^{-s^2}\, ds - \frac{1}{a\sqrt{\pi t}} \int_{\alpha}^{\infty} e^{-\left[a\tau-\alpha a+\frac{\tau^2}{4t}\right]}\, d\tau$$

$$= \frac{1}{a}\operatorname{Erf}\left(\frac{\alpha}{2\sqrt{t}}\right) - \frac{1}{a\sqrt{\pi t}} \int_{\alpha}^{\infty} e^{a\alpha+a^2t} e^{-\left(\frac{\tau}{2\sqrt{t}}+a\sqrt{t}\right)^2} d\tau$$

$$= \frac{1}{a}\operatorname{Erf}\left(\frac{\alpha}{2\sqrt{t}}\right) - \frac{e^{a(at+\alpha)}}{a}\frac{2}{\sqrt{\pi}} \int_{\frac{\alpha}{2\sqrt{t}}+a\sqrt{t}}^{\infty} e^{-z^2}\, dz,$$

where $z = \tau/2\sqrt{t} + a\sqrt{t}$. Thus,

$$\frac{e^{-\alpha\sqrt{p}}}{p(a+\sqrt{p})} \doteqdot \frac{1}{a}\operatorname{Erf}\left(\frac{\alpha}{2\sqrt{t}}\right) - \frac{e^{a(at+\alpha)}}{a}\operatorname{Erf}\left(\frac{\alpha}{2\sqrt{t}}+a\sqrt{t}\right).$$

643. *Solution.* $I(t) = \dfrac{1}{\sqrt{\pi t}} \displaystyle\int_0^{\infty} \cosh\tau e^{-\tau^2/4t}\, d\tau.$

Comparing $I(t)$ and (14.19), we see that $f(t) = \cosh t$, which means that $F(p) = p/(p^2-1)$. Consequently, $F(\sqrt{p}) = \sqrt{p}/(p-1)$. Putting $\Phi(p) = 1/\sqrt{p}$, we find that $\Phi(p)\, F(\sqrt{p}) = 1/(p-1) \doteqdot I(t)$; whence $I(t) = e^t$.

644. $I(t) = e^{-t}$. **645.** $I(t) = 2te^t$. **646.** $I(t) = 2te^{-t}$.

647. $x(t) = (t+1)e^{-t}$. **648.** $x(t) = -1$.

649. $x(t) = \dfrac{e^{-2t} - \cos t + 2\sin t}{5}$. **650.** $x(t) = t + \dfrac{1}{2}t^2$.

651. $x(t) = t$. **652.** $x(t) = \cos t$.

653. $x(t) = \dfrac{1}{4}e^t + \dfrac{5}{12}e^{-3t} - \dfrac{2}{3}$.

654. $x(t) = \dfrac{1}{4}(1 - e^{2t} + 2te^{2t})$.

655. $x(t) = \dfrac{1}{8}(3e^t - e^{-3t} - 2e^{-t})$. **656.** $x(t) = t - \sin t$.

657. $x(t)=\frac{2}{25}e^{-2t}-\frac{2}{25}\cos t+\frac{14}{25}\sin t-\frac{1}{5}t\sin t-\frac{2}{5}t\cos t.$

658. $x(t)=\frac{1}{2}(e^{-t}-te^{-t}-\cos t)$

659. $x(t)=\frac{1}{2}e^{t}-t-1+\frac{1}{2}(\cos t+\sin t).$

660. $x(t)=\frac{1}{2}t^{2}-1+\cos t-\sin t.$

661. $x(t)=\frac{1}{2}t^{2}e^{t}+te^{t}.$

662. $x(t)=\frac{3}{5}e^{-t}\sin 2t-\frac{4}{5}e^{-t}\cos 2t-\frac{1}{5}.$

663. $x(t)=\frac{1}{2}(1-e^{t}\cos t+e^{t}\sin t).$

664. $x(t)=2+\frac{1}{2}(e^{-t}-\cos t+\sin t).$

665. $x(t)=t^{2}-4t+6-5e^{-t}-te^{-t}.$

666. $x(t)=2t+\frac{1}{2}(e^{-t}+\cos t-\sin t).$

667. $x(t)=\frac{1}{2}t\sin t-\cos t+\sin t.$

668. $x(t)=\frac{1}{6}t^{3}-\frac{1}{2}t^{2}+2t-4+e^{-t}.$

669. $x(t)=\frac{3}{5}+\frac{2}{5}e^{-t}\cos 2t+\frac{1}{5}e^{-t}\sin 2t.$

670. $x(t)=\frac{1}{2}(\cos t+\cosh t)-t-1.$

671. $x(t)=\frac{1}{2}(1-e^{-t}\cos t-e^{-t}\sin t).$

672. $x(t)=1-2\cos t.$

673. $x(t)=\frac{1}{4}t+\cos 2t-\frac{1}{8}\sin 2t.$

674. $x(t)=\frac{3}{25}-\frac{t}{5}-\frac{3}{25}e^{t}\cos 2t+\frac{4}{25}e^{t}\sin 2t.$

675. $x(t)=e^{-t}-e^{t/2}\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}e^{t/2}\sin\frac{\sqrt{3}}{2}t$

676. $x(t)=-1-\frac{1}{2}(\sin t+\cos t+e^{-t}).$

677. $x(t)=\frac{1}{2}e^t+\frac{3}{2}\sin t+\frac{1}{2}\cos t-1$.

678. $x(t)=\cosh t-\frac{1}{2}t^2-1$.

679. $x(t)=2+\frac{1}{2}(e^t+\sin t-\cos t)$.

680. $x(t)=e^t\left(1-t+\frac{1}{2}t^2\right)-1$.

681. $x(t)=-\frac{3}{2}\sin t-\frac{1}{2}t\cos t$.

682. $x(t)=2e^{-t}+te^{-t}+t-2$.

683. $x(t)=\frac{1}{3}e^{-t}-\frac{1}{3}e^{t/2}\left(\cos\frac{\sqrt{3}}{2}t-3\sqrt{3}\sin\frac{\sqrt{3}}{2}t\right)$.

684. $x(t)=-\frac{1}{4}e^t-\frac{3}{4}e^{-t}-\frac{1}{2}\sin t$.

685. $x(t)=\frac{1}{2}e^t-\frac{5}{6}e^{-t}+\frac{1}{3}e^{t/2}\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}e^{t/2}\sin\frac{\sqrt{3}}{2}t$.

686. $x(t)=\cos t-t\cos t$.

687. $x(t)=2+t-\frac{1}{2}\cos t+\frac{1}{2}te^t-\frac{3}{2}e^t$.

688. $x(t)=1-\frac{22}{25}e^{-t}-\frac{6}{5}te^{-t}-\frac{3}{25}\cos 2t+\frac{4}{25}\sin 2t$.

689. $x(t)=\frac{t}{4}\sin 2t+\frac{1}{12}(\cos 2t-\cos 4t)$.

690. $x(t)=\frac{1}{2}(t-1)e^t+\frac{1}{2}\cos t+2\sin t-2t\cos t$.

691. $x(t)=e^t\left(\frac{t^2}{2}-t+1\right)$.

692. $x(t)=2t-3+3e^{-t}-\frac{1}{5}(\sin 2t-2\cos 2t+2e^{-t})$.

693. $x(t)=4t+3-2e^t$. 694. $x(t)=e^{2t}-e^t-te^t$.

695. $x(t)=3e^t-3-2t-t^2-\frac{t^3}{3}$.

696. $x(t)=\frac{1}{4}e^t\left(t^2-3t+\frac{3}{2}\right)+\frac{1}{3}e^{\frac{t}{2}}\left(\sqrt{3}\sin\frac{\sqrt{3}}{2}t-\right.$
$\left.-\cos\frac{\sqrt{3}}{2}t\right)-\frac{1}{24}e^{-t}$.

697. $x(t)=\frac{4}{9}\sin 2t-\frac{5}{9}\sin t-\frac{1}{3}t\cos 2t.$

698. $x(t)=\frac{a}{2n^2}[\sin nt\cos\alpha-nt\cos(nt+\alpha)].$

699. $x(t)=\frac{1}{6}t^2-\frac{4}{9}t+\frac{35}{54}-e^{-t}+\frac{1}{2}e^{-2t}-\frac{4}{27}e^{-3t}.$

700. $x(t)=-\frac{t}{24}[3t\cos t+(t^2-3)\sin t].$

701. $x(t)=\frac{1}{\beta}e^{\alpha t}\sin\beta t.$

702. $x(t)=\frac{1}{3}\sin t-\frac{1}{6}\sin 2t.$

703. $x(t)=\frac{1}{10}e^{2t}-\frac{1}{2}+\frac{2}{5}\cos t-\frac{1}{5}\sin t.$

704. $x(t)=\frac{\gamma}{2}t^2+(1-\gamma)t+(\gamma-1)+\left(\frac{1}{2}-\gamma\right)e^{-t}+\frac{1}{2}(\cos t-\sin t).$

705. $x(t)=\frac{83}{80}\cosh 2t-\frac{1}{10}\cos t+\frac{1}{16}\cos 2t.$

706. $x(t)=e^t\left(\cos t+\sin t-\frac{1}{2}\right)+\frac{1}{2}e^{-3t}.$

707. $x(t)=\frac{1}{3}e^{-t/2}\left(\cos\frac{\sqrt{3}}{2}t+\frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t\right)+\frac{1}{3}(t-1)e^t.$

708. $x(t)=\frac{1}{4}(t^2\sin t+t\cos t-\sin t).$

709. $x(t)=\frac{2}{9}[e^t-e^{-3t}(3t+1)].$

710. $x(t)=1-e^{-t}\left(\frac{t^2}{2}+t+1\right).$

711. $x(t)=1-\frac{1}{3}e^{-t}-\frac{2}{3}e^{t/2}\cos\frac{\sqrt{3}}{2}t.$

712. $x(t)=\frac{2a}{\omega^2}\left[\sin^2\frac{\omega t}{2}\eta(t)-\sin^2\frac{\omega(t-b)}{2}\eta(t-b)\right].$

713. $x(t)=-\cos t.$ 714. $x(t)=(t-1)^2+e^{1-t}.$

715. $x(t)=t^2+2t.$ 716. $x(t)=\left(t-1-\frac{\pi}{2}\right)\cos t.$

717. $x(t)=(t^2-2t+2)e^{1-t}.$

718. $x(t)=\frac{1}{2}\left(t-\frac{1}{2}\sin 2t\right)\eta(t)-\left[(t-1)-\frac{1}{2}\sin 2(t-1)\right]$
$\times\eta(t-1)+\frac{1}{2}\left[(t-2)-\frac{1}{2}\sin 2(t-2)\right]\eta(t-2).$

719. $x(t)=[b+(1-b)\cos t]\,\eta(t)+[b-b\cos(t-a)]\,\eta(t-a).$

720. $x(t)=\frac{1}{3}\sin 3t\,\eta(t)+\frac{1}{9}\left[(t-1)-\frac{1}{3}\sin 3(t-1)\right]\eta(t-1)$
$-\frac{2}{9}\left[(t-2)\frac{1}{3}\sin 3(t-2)\right]\eta(t-2)$
$+\frac{1}{9}\left[(t-3)-\frac{1}{3}\sin 3(t-3)\right]\eta(t-3).$

721. $x(t)=\sum_{k=0}^{3}(-1)^k\left[1-e^{t-ka}+e^{t-ka}(t-ka)\right]\eta(t-ka).$

722. *Solution.* Newton's second law is $m\ddot{x}=-m\lambda\dot{x}-2m\mu x$; $x(0)=x_0$, $\dot{x}(0)=v_0$. The operator equation is

$$p^2X-px_0-v_0+2\mu pX-2\mu x_0+\lambda X=0,$$

whence $X(p)=\dfrac{2\mu x_0+v_0+px_0}{p^2+2\mu p+\lambda}$, or

$$X(p)=\frac{x_0(p+\mu)+\mu x_0+v_0}{(p+\mu)^2+(\sqrt{\lambda-\mu^2})^2}=\frac{x_0(p+\mu)}{(p+\mu)^2+n^2}+\frac{\mu x_0+v_0}{(p+\mu)^2+n^2},$$

where $n^2=\lambda-\mu^2$. The object function for $X(p)$ is

$$x(t)=\frac{1}{n}e^{-\mu t}\left[nx_0\cos nt+(\mu x_0+v_0)\sin nt\right].$$

723. Newton's second law is
$m\ddot{x}=-mn^2x+F\eta(t)-F\eta(t-T)$, with $x(0)=0$ and $\dot{x}(0)=0$.

724. Newton's second law is $\ddot{x}=an^2-n^2x$, $x(0)=0$, $\dot{x}(0)=0$.

725. Newton's second law is
$m\ddot{x}=-mg-2km\dot{x}$, with $x(0)=0$, and $\dot{x}(0)=v_0$.

726. *Solution.* Newton's second law is

$$m\ddot{x}=F. \tag{1}$$

In our case $m=2$ and $F=F_0+at=4+at$, so that Eq. (1) takes the form

$$2\ddot{x}=4+at, \tag{2}$$

$$x(0)=0, \quad \dot{x}(0)=10. \tag{3}$$

The operator equation is $2(p^2X - 10) = 4/p + a/p^2$, whence $X = \frac{1}{p^2}\left(\frac{a}{2p^2} + \frac{2}{p} + 10\right)$. The object function for $X(p)$ is

$$x(t) = \frac{a}{12}t^3 + t^2 + 10t.$$

To find a we have the following system of equations:

$$\begin{cases} 450 = \frac{at_0^3}{12} + t_0^2 + 10t_0, \\ 105 = \frac{at_0^2}{4} + 2t_0 + 10, \end{cases}$$

which yields $t_0 = 10$ and $a = 3$.

727. Newton's second law is

$$m\ddot{x} = 4mx - 3m\dot{x}, \text{ with } x(0) = 1 \text{ and } \dot{x}(0) = 0.$$

The equation of motion is $x(t) = \frac{1}{5}(4e^t + e^{-4t})$.

728. Newton's second law is $m\ddot{x} = mg - \lambda\dot{x}$. Since we know that $\lambda = mg/3$ at $v = 1$ m/s, the final law of motion is

$$\frac{dv}{dt} = g - \frac{1}{3}gv, \text{ with } v(0) = 0;$$

whence $v(t) = 3\left(1 - e^{-\frac{gt}{3}}\right)$, and $v_{max} = 3$ at $t = \infty$.

729. Newton's second law is $m\ddot{x} = -k\dot{x}$, with $x(0) = 0$ and $\dot{x}(0) = v_0$.

The equation of motion is $x(t) = \frac{mv_0}{k}\left(1 - e^{-\frac{k}{m}t}\right)$, and $x_{max} = \frac{mv_0}{k}$ at $t = \infty$.

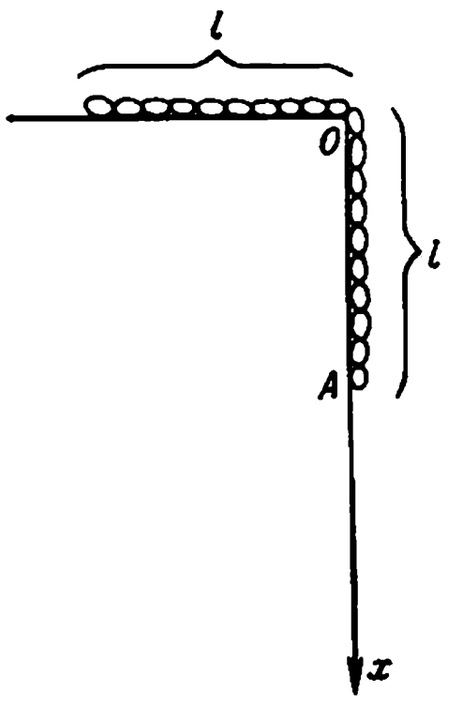


Figure to answer to Problem 730

730. *Solution.* Let us describe the motion of the lower end of the chain. We select the origin of coordinates at point O (see the figure) and send the x axis downward. Then the initial conditions are $x(0) = l$ and $\dot{x}(0) = 0$ (the chain is at rest). If the abscissa of the lower end of the chain is x, the driving force is the weight of the part of the chain

that is hanging off the side of the table, i.e.

$$F = \frac{mg}{2l} x.$$

The differential equation of the chain's motion is

$$m\ddot{x} = \frac{mg}{2l} x, \quad \text{with} \quad x(0) = l \text{ and } \dot{x}(0) = 0,$$

and $x(t) = \frac{l}{2}\left(e^{t\sqrt{\frac{g}{2l}}} + e^{-t\sqrt{\frac{g}{2l}}}\right)$. This is the equation of motion of the chain moving up to the time T when it slips off the table. We can find T if we put $x = 2l$:

$$4 = e^{T\sqrt{\frac{g}{2l}}} + e^{-T\sqrt{\frac{g}{2l}}}.$$

Introducing the notation $z = e^{T\sqrt{g/2l}}$, we arrive at the equation

$$z^2 - 4z + 1 = 0,$$

from which it follows that $z_1 = 2 - \sqrt{3} < 1$ and $z_2 = 2 + \sqrt{3}$; we drop z_1 since it corresponds to a negative T. Thus, we have the following equation for T:

$$e^{T\sqrt{\frac{g}{2l}}} = 2 + \sqrt{3}, \quad \text{whence} \quad T = \sqrt{\frac{2l}{g}} \ln (2 + \sqrt{3}).$$

731. Newton's second law is

$$m\ddot{x} = -2mxk^2, \text{ with } x(0) = a \text{ and } \dot{x}(0) = 0,$$

and the equation of motion is $x(t) = a \cos(\sqrt{2}\, kt)$.

732. Newton's second law is

$$m\ddot{x} = -\mu m x, \text{ with } x(0) = a \text{ and } \dot{x}(0) = 0,$$

and $t = \frac{\pi}{2\sqrt{\mu}}$.

733. Newton's second law is

$$m\ddot{x} = -k\dot{x}, \text{ with } x(0) = 0 \text{ and } \dot{x}(0) = 6,$$

and $x(t) = 600(1 - e^{-0.01t})$.

734. Newton's second law is

$$\ddot{x} + 4x = 2\cos t, \text{ with } x(0) = 0 \text{ and } \dot{x}(0) = 0,$$

and $x(t) = \frac{2}{3}(\cos t - \cos 2t)$.

735. Newton's second law is $m\ddot{r} = -mk^2 r$, or

$$\ddot{x} = -k^2 x, \text{ with } x(0) = a \text{ and } \dot{x}(0) = 0;$$
$$\ddot{y} = -k^2 y, \text{ with } y(0) = 0 \text{ and } \dot{y}(0) = v_0.$$

The point follows an ellipse, $\frac{x^2}{a^2} + \frac{y^2}{\frac{v_0^2}{k^2}} = 1$.

736. Newton's second law is

$$\ddot{x} = k^2 x, \text{ with } x(0) = a \text{ and } \dot{x}(0) = 0;$$

$$\ddot{y} = k^2 y, \text{ with } y(0) = 0 \text{ and } \dot{y}(0) = v_0.$$

The particle moves along a hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{\frac{v_0^2}{k^2}} = 1$.

737. ODE: $L\frac{d^2Q}{dt^2} + \frac{1}{C}Q = E\cos(\omega t + \alpha)$, $Q|_{t=0} = 0$, $\left.\frac{dQ}{dt}\right|_{t=0} = 0$.

738. ODE: $L\frac{d^2Q}{dt^2} + \frac{1}{C}Q = E\sin nt$, $Q|_{t=0} = 0$, $\left.\frac{dQ}{dt}\right|_{t=0} = 0$.

739. ODE: $L\frac{dI}{dt} + RI = E\sin(\omega t + \alpha)$, $I|_{t=0} = 0$.

740. ODE: $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E_1\eta(t) + (E_2 - E_1)\eta(t-T)$,

$$Q|_{t=0} = 0, \quad \left.\frac{dQ}{dt}\right|_{t=0} = 0.$$

741. ODE: $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E$, $Q|_{t=0} = 0$, $\left.\frac{dQ}{dt}\right|_{t=0} = 0$.

742. $x(t) = (C_1 + C_2 t^2)e^{-t}$. 743. $x(t) = C_1$.

744. $x(t) = e^{-t}$. 745. $x(t) \equiv -1$. 746. $x(t) = e^t$.

747. (a) $x(t) = \sum_{s=0}^{k} (-1)^s 2^s s! \binom{k}{s} \frac{t^{2s}}{(2s)!}$;

(b) $x(t) = \sum_{s=0}^{k} (-1)^s 2^s s! \binom{k}{s} \frac{t^{2s+1}}{(2s+1)!}$.

748. $x(t) = \frac{1}{2}(e^t - 1) - \ln\frac{1+e^t}{2}$.

749. $x(t) = e^{-t}[(t+1)\ln(t+1) - t]$.

750. $x(t) = (e^t + 2)\ln\frac{e^t+2}{3} - e^t + 1$.

751. $x(t) = e^t - 1 - (t + \ln 2)(e^t + 1) + (e^t + 1)\ln(e^t + 1)$.

752. $x(t) = \sin t\left(t - \frac{4}{\sqrt{3}}\arctan\frac{\tan\frac{t}{2}}{\sqrt{3}}\right) + \cos t\ln(2+\cos t) - \ln 3\cos t$.

753. $x(t)=\frac{1}{3}-\frac{9-\pi\sqrt{3}}{27}\cos t+\frac{\sqrt{3}}{36}\sin t\ln\left|\frac{\sqrt{3}\sin t-2}{\sqrt{3}\sin t+2}\right|$
$-\frac{\sqrt{3}}{9}\cos t\arctan(\sqrt{3}\cos t)$.

754. $x(t)=\cos t\arctan(\cos t)-\frac{\pi}{4}\cos t-\frac{1}{2\sqrt{2}}\sin t$
$\times\ln\left|\frac{\sin t-\sqrt{2}}{\sin t+\sqrt{2}}\right|$.

755. $x(t)=\sin t\arctan(\sin t)+\cos t\frac{1}{2\sqrt{2}}\left\{\ln\left|\frac{\sqrt{2}+\cos t}{\sqrt{2}-\cos t}\right|\right.$
$\left.-\ln(3+2\sqrt{2})\right\}$.

756. $x(t)=-\sinh t+2\cosh t\left(\arctan e^t-\frac{\pi}{4}\right)$.

757. $x(t)=\ln 2\cos t-\cos t\ln(2+\sin t)-t\sin t+\frac{2}{\sqrt{3}}(2\sin t$
$+1)\left(\arctan\frac{2\tan\frac{t}{2}-1}{\sqrt{3}}-\frac{\pi}{6}\right)$.

758. $x(t)=e^t$, $y(t)=-e^t$.
759. $x(t)=e^t$, $y(t)=e^t$.
760. $x(t)=2(1-e^{-t}-te^{-t})$, $y(t)=2-t-2e^{-t}-2te^{-t}$.

761. $x(t)=\frac{1}{4}(e^t-e^{3t}+2te^{3t})$, $y(t)=\frac{1}{4}(5e^t-e^{3t}-2te^{3t})$.

762. $x(t)=e^t(\cos t-2\sin t)$, $y(t)=e^t(\cos t+3\sin t)$.

763. $x(t)=\frac{1}{3}(e^t+2\cos 2t+\sin 2t)$,

$y(t)=\frac{2}{3}\left(e^t-\cos 2t-\frac{1}{3}\sin 2t\right)$.

764. $x(t)=e^t-\frac{11}{34}e^{4t}-\frac{3}{17}\cos t+\frac{5}{17}\sin t-\frac{1}{2}$,

$y(t)=-\frac{2}{3}e^t+\frac{22}{51}e^{4t}+\frac{4}{17}\cos t-\frac{1}{17}\sin t$.

765. $x(t)=-\frac{1}{15}e^{-2t}+\frac{13}{12}e^{-t}-2+\frac{1}{6}e^t+\frac{2}{3}e^{2t}+\frac{3}{20}e^{3t}$.

$y(t)=\frac{1}{15}e^{-2t}+\frac{13}{12}e^{-t}-2-\frac{1}{6}e^t+\frac{2}{3}e^{2t}+\frac{7}{20}e^{3t}$,

$$z(t)=-\frac{13}{12}e^{t}-\frac{1}{2}e^{t}+\frac{4}{3}e^{2t}+\frac{1}{4}e^{3t}.$$

766. $x(t)=-e^{t}$, $y(t)=0$, $z(t)=e^{t}$.

767. $x(t)=\frac{2}{5}(e^{3t}-e^{-2t})$, $y(t)=\frac{1}{5}(3e^{3t}+2e^{-2t})$,

$$z(t)=\frac{1}{5}(3e^{3t}+2e^{-2t}).$$

768. $x(t)=\frac{3e^{-2t}}{4(2+a)}+\frac{(11-4a)e^{2t}}{4(2-a)}+\frac{3e^{at}}{a^2-4}$.

$$y(t)=-\frac{e^{-2t}}{4(2+a)}+\frac{(11-4a)e^{2t}}{4(2-a)}+\frac{(a-1)e^{at}}{a^2-4}.$$

769. $x(t)=2-e^{-t}$, $y(t)=2-e^{-t}$, $z(t)=2e^{-t}-2$.

770. $x(t)=6e^{t}-e^{2t}-4e^{3t}$, $y(t)=3e^{t}-2e^{3t}$,

$$z(t)=6e^{3t}+e^{2t}-6e^{t}.$$

771. $x(t)=-\frac{1}{15t^2}+\frac{13}{12t}-2+\frac{1}{6}t+\frac{2}{3}t^2-\frac{3}{20}t^3$.

$$y(t)=\frac{1}{15t^2}+\frac{13}{12t}-2-\frac{1}{6}t+\frac{2}{3}t^2+\frac{7}{20}t^3,$$

$$z(t)=-\frac{13}{12t}-\frac{1}{2}t+\frac{4}{3}t^2+\frac{1}{4}t^3.$$

772. $x_m(t)=e^{-ct}\frac{(ct)^m}{m!}$ $(m=0, 1, 2, \ldots, n)$.

773. $x(t)=\frac{1}{2}-\frac{1}{5}e^{-t}-\frac{3}{10}e^{-6t/11}$, $y(t)=\frac{1}{5}(e^{-t}-e^{-6t/11})$.

774. $x(t)=\frac{1}{3}+\frac{2}{3}t$, $y(t)=\frac{5}{4}t^2-\frac{1}{4}$,

$$z(t)=\frac{5}{4}t^2-\frac{2}{3}t+\frac{5}{12}.$$

775. $x(t)=\frac{28}{9}e^{3t}-e^{-t}-\frac{t}{3}-\frac{1}{9}$,

$$y(t)=\frac{28}{9}e^{3t}+e^{-t}-\frac{t}{3}-\frac{1}{9}.$$

776. Newton's second law is

$$\begin{cases} m\ddot{x}=-\frac{eH}{c}\dot{y}, & x(0)=0, \quad \dot{x}(0)=v_0, \\ m\ddot{y}=\frac{eH}{c}\dot{x}, & y(0)=0, \quad \dot{y}(0)=0. \end{cases}$$

The equation of motion is $x(t)=\frac{v_0 mc}{eH}\sin\frac{eHt}{mc}$ and $y(t)=\frac{mcv_0}{eH}\times$ $\times\left(1-\cos\frac{eHt}{mc}\right)$. The electron moves along a path given by the following equation:

$$x^2+y^2-\frac{2mcv_0}{eH}y=0.$$

777. Newton's second law of motion of the projectile is

$$\begin{cases} m\ddot{x}=0, & x(0)=0, \quad \dot{x}(0)=\frac{v_0}{\sqrt{2}}, \\ m\ddot{y}=-gm, & y(0)=0, \quad \dot{y}(0)=\frac{v_0}{\sqrt{2}}. \end{cases}$$

The maximum height is $H=v_0^2/4g$ and the range is $x=v_0^2/g$.

778. Suppose the electron is emitted at the origin of coordinates, O. We direct the x axis in such a way that it is parallel to the magnetic field H and the y axis in such a way that the electron's velocity $\mathbf{v}_0$ lies in the xOy plane. Newton's second law is

$$\begin{cases} m\ddot{x}=0, & x(0)=0, \quad \dot{x}(0)=v_0\cos\alpha, \\ m\ddot{y}=-\frac{eH}{c}\dot{z}, & y(0)=0, \quad \dot{y}(0)=v_0\sin\alpha, \\ m\ddot{z}=\frac{eH}{c}\dot{y}, & z(0)=0, \quad \dot{z}(0)=0. \end{cases}$$

The path that the electron follows is given by the equations

$$\begin{cases} y^2+z^2-\frac{2v_0 cm\sin\alpha}{eH}z=0, \\ x=tv_0\cos\alpha. \end{cases}$$

779. Newton's second law is

$$\begin{cases} m\ddot{x}=-km\dot{x}, & x(0)=0, \quad \dot{x}(0)=v_0\cos\alpha, \\ m\ddot{y}=-mg-km\dot{y}, & y(0)=0, \quad \dot{y}(0)=v_0\sin\alpha, \end{cases}$$

The equation of motion is $x(t)=\frac{v_0\cos\alpha}{k}(1-e^{-kt})$ and $y(t)=\frac{v_0 k\sin\alpha+g}{k^2}(1-e^{-kt})-\frac{gt}{k}$.

780. Newton's second law is

$$\begin{cases} m\ddot{x}=-\frac{eH}{c}\dot{y}-km\dot{x}, & x(0)=0, \quad \dot{x}(0)=u, \\ m\ddot{y}=\frac{eH}{c}\dot{x}-km\dot{y}, & y(0)=0, \quad \dot{y}(0)=0, \\ m\ddot{z}=-kmz, & z(0)=0, \quad \dot{z}(0)=0. \end{cases}$$

781. Newton's second law is

$$\begin{cases} m\ddot{x} = -2\lambda\dot{x} - \mu^2 x, & x(0) = a, \quad \dot{x}(0) = 0, \\ m\ddot{y} = -2\lambda\dot{y} - \mu^2 y, & y(0) = 0, \quad \dot{y}(0) = v_0, \end{cases}$$

782. Newton's second law is

$$\begin{cases} m\ddot{x} = 0, & x(0) = 0, \quad \dot{x}(0) = v_0, \\ m\ddot{y} = -mk^2 y, & y(0) = a, \quad \dot{y}(0) = 0. \end{cases}$$

The equation of motion is $x(t) = v_0 t$ and $y(t) = a \cos kt$. The point follows a path given by the equation

$$y = a \cos\left(\frac{kx}{v_0}\right).$$

783. $\varphi(x) = \frac{1}{2} \sinh x + \frac{1}{2} \sin x.$

784. $\varphi(x) = \frac{1}{3}\left(e^x - e^{-x/2} \cos \frac{\sqrt{3}\,x}{2} + \sqrt{3}\, e^{-x/2} \sin \frac{\sqrt{3}\,x}{2}\right).$

785. $\varphi(x) = x + \frac{1}{6} x^3.$

786. $\varphi(x) = \frac{2}{5} e^{2x} + \frac{3}{5} \cos x + \frac{1}{5} \sin x.$

787. $\varphi(x) = 2 + x - e^{\frac{x}{2}}\left(\cos \frac{\sqrt{3}\,x}{2} - \sqrt{3} \sin \frac{\sqrt{3}\,x}{2}\right).$

788. $\varphi(x) = -\frac{1}{16} - \frac{1}{8} x + \frac{3}{8} x^2 + \frac{1}{16} e^{2x} - \frac{1}{12} x^3.$

789. $\varphi(x) = \frac{1}{2} e^{-x} + \frac{1}{6} e^x$

$$+ \frac{1}{3} e^{-x/2}\left(\cos \frac{\sqrt{3}\,x}{2} - \sqrt{3} \sin \frac{\sqrt{3}\,x}{2}\right).$$

790. $\varphi(x) = \frac{1}{3}\left(e^x - e^{-x} + \frac{\sqrt{2}}{2} \sin \sqrt{2}\,x\right).$

791. $\varphi(x) = xe^x.$

792. $\varphi(x) = e^x.$ 793. $\varphi(x) = \frac{4}{3} - \frac{1}{3} \cos \sqrt{3}\,x.$

794. $\varphi(x) = \cosh x - xe^{-x}.$ 795. $\varphi(x) = \frac{1}{2}(\cosh x + \cos x).$

796. $\varphi(x) = x - \frac{1}{6} x^3.$ 797. $\varphi(x) = \frac{2}{\sqrt{5}} \sinh \frac{\sqrt{5}}{2} x.$

798. $\varphi(x)=1-x$.

799. $\varphi(x)=J_0(x)$, which yields $\int\limits_0^x J_0(x-t)\,J_0(t)\,dt=\sin x$.

800. $\varphi(x)\equiv 1$. 801. $\varphi(x)=e^{-x}$.

802. $\varphi(x)=1+2x+\frac{1}{2}x^2+\frac{1}{3}x^3$. 803. $\varphi(x)=2xe^x-x^2e^x$.

804. $\varphi(x)\equiv 1$. 805. $\varphi(x)=1-\frac{x^2}{2}$.

806. $\varphi_1(x)=e^{-x}(1-x)$, $\varphi_2(x)=\frac{8}{9}e^{2x}+\frac{1}{3}xe^{-x}-\frac{8}{9}e^{-x}$.

807. $\varphi_1(x)=e^{2x}$, $\varphi_2(x)=\frac{1}{2}-\frac{1}{2}e^{2x}$.

808. $\varphi_1(x)=\frac{1}{3}e^{3/2x}\left(\sqrt{3}\sin\frac{\sqrt{3}}{2}x+2\cos\frac{\sqrt{3}}{2}x\right)-\frac{1}{3}$,

$\varphi_2(x)=e^{3/2x}\left(\cos\frac{\sqrt{3}}{2}x-\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x\right)$.

809. $\varphi_1(x)=(x+2)\sin x+(2x+1)\cos x$,

$\varphi_2(x)=\left(1+\frac{x}{2}\right)\cos x-\left(\frac{1}{2}+x\right)\sin x$.

810. $\varphi_1(x)=\frac{\sqrt{3}}{2}\sin\sqrt{3}\,x-\frac{3}{2}\sinh x$,

$\varphi_2(x)=\cos\sqrt{3}\,x-3\cosh x$.

811. $\varphi_1(x)=2(1-x)e^{-x}$, $\varphi_2(x)=(1-x)e^{-x}$.

812. $x(t)=\sum\limits_{k=0}^{\infty}\frac{(t-k)^{2k+3}}{(2k+3)!}\eta(t-k)$.

813. $x(t)=\sum\limits_{k=0}^{\infty}\frac{2^k(t-k)^{k+3}}{(k+3)!}\eta(t-k)$.

814. $x(t)=\sum\limits_{k=0}^{\infty}\frac{(k+1)(t-k)^{k+2}}{(k+2)!}\eta(t-k)$.

815. $x(t)=\sum\limits_{k=0}^{\infty}(-1)^k\frac{(t-2k)^{k+3}(k+1)}{(k+3)!}\eta(t-2k)$.

816. $x(t)=\left(-t+\frac{1}{2}t^2\right)\eta(t)$

$$+\sum_{k=3}^{\infty}\frac{(t-k+2)^{k+1}}{k!}(t-3k+2)\,\eta(t-k+2).$$

817. $x(t)=\left(1+t+\frac{t^2}{2!}\right)\eta(t)$

$$+\sum_{k=1}^{\infty}\frac{(t-k)^{k+1}}{(k+1)!}\eta(t-k)+\sum_{k=1}^{\infty}\frac{(t-k)^{k+2}}{(k+2)!}\eta(t-k).$$

818. $x(t)=\cos t.$

819. $u(x,\ t)=u_0\left(1-\frac{2}{\sqrt{\pi}}\int\limits_0^{\frac{x}{2\sqrt{kt}}}e^{-z^2}\,dz\right).$

820. $u(x,\ t)=u_1\frac{2}{\sqrt{\pi}}\int\limits_0^{\frac{x}{2\sqrt{kt}}}e^{-z^2}\,dz.$

821. $u(x,\ t)=ae^{-x\sqrt{\omega/2k}}\cos(\omega t-x\sqrt{\omega/2k})$

$$-\frac{a}{\pi}\int_0^{\infty}e^{-\rho t}\sin x\sqrt{\frac{\rho}{k}}\frac{\rho\,d\rho}{\rho^2+\omega^2}.$$

822. $u(x,t)=a\left[e^{-x\sqrt{\frac{\omega}{2k}}}\sin\left(\omega t-x\sqrt{\frac{\omega}{2k}}\right)\right.$

$$\left.+\frac{\omega}{\pi}\int_0^{\infty}e^{-\rho t}\sin x\sqrt{\frac{\rho}{k}}\frac{d\rho}{\rho^2+\omega^2}\right].$$

823. $u(x,\ t)=\frac{x}{2\sqrt{\pi k}}\int\limits_0^t\varphi(\tau)\frac{e^{-\frac{x^2}{4k(t-\tau)}}}{(t-\tau)^{3/2}}\,d\tau.$

824. PDE: $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$, $0\leqslant x\leqslant l$, $t>0$,

$$u\,|_{t=0}=u_0,\quad \left.\frac{\partial u}{\partial x}\right|_{x=0}=0,\quad u\,|_{x=l}=u_1,$$

$$u(x, t)=u_1+\frac{4(u_1-u_0)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-a^2\left(n-\frac{1}{2}\right)^2\frac{\pi^2}{l^2}t}\cos\frac{(2n-1)\pi x}{2l}.$$

825. PDE: $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$, $x>0$, $t>0$,

$$u|_{t=0}=u_0,\ \left.\frac{\partial u}{\partial x}\right|_{x=0}=hu|_{x=0},\quad h=\text{const},$$

$$u(x, t)=u_0\left[\operatorname{erf}\frac{x}{2a\sqrt{t}}+e^{hx+h^2a^2t}\operatorname{Erf}\left(\frac{x}{2a\sqrt{t}}+ha\sqrt{t}\right)\right].$$

Hint. Use Êfros's theorem (see § 14).

826. PDE: $\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$, $0\leqslant x\leqslant l$, $t>0$,

$$u|_{t=0}=0,\quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=0,\quad u(0, t)=0,\quad \frac{\partial u(l, t)}{\partial x}=\frac{F}{E},$$

$$u(x, t)=\frac{Fx}{E}-\frac{8Fl}{\pi^2E}\sum_{k=0}^{\infty}\frac{(-1)^k\sin\frac{(2k+1)\pi x}{2l}\cos\frac{(2k+1)\pi ct}{2l}}{(2k+1)^2}.$$

827. PDE: $\frac{\partial^2 u}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\frac{g}{c^2}$, $0\leqslant x\leqslant l$, $t>0$,

$$u|_{t=0}=0,\quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=0,\quad u(0, t)=0,\quad \frac{\partial u(l, t)}{\partial x}=0,$$

$$u(x, t)=\frac{gx(2l-x)}{2c^2}$$

$$-\frac{16gl^2}{\pi^3c^2}\sum_{k=0}^{\infty}(-1)^k\frac{\cos\frac{(2k+1)(x-l)\pi}{2l}\cos\frac{(2k+1)\pi ct}{2l}}{(2k+1)^3}.$$

828. $u(x, t)=-\frac{bx}{12}(x^3-2lx^2+l^3)$

$$+\frac{8bl^4}{\pi^5}\sum_{k=0}^{\infty}\frac{\cos\frac{(2k+1)\pi t}{l}\sin\frac{(2k+1)\pi x}{l}}{(2k+1)^5}.$$

829. PDE: $\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}$, $0 \leqslant x \leqslant l$, $t > 0$,

$u(x, 0) = \dfrac{4hx(l-x)}{l^2}$, $\dfrac{\partial u(x, 0)}{\partial t} = 0$, $u(0, t) = u(l, t) = 0$.

$$u(x, t) = \frac{32h}{\pi^3} \sum_{k=0}^{\infty} \frac{\cos \dfrac{(2k+1)\pi a t}{l} \sin \dfrac{(2k+1)\pi x}{l}}{(2k+1)^3}.$$

830. $\dfrac{1}{1-e^{-p}}$. 831. $\dfrac{e^{-p}}{(1-e^{-p})^2}$. 832. $\dfrac{1}{1-e^{\alpha-p}}$.

833. $\dfrac{e^p}{e^p-\alpha}$. 834. $\dfrac{e^p(e^p-\cos 1)}{e^{2p}+2e^p\cos 1+1}$.

835. $\dfrac{e^p \sin \alpha}{e^{2p}-2e^p\cos\alpha+1}$. 836. $\dfrac{e^p \sinh 1}{e^{2p}-2e^p\cosh 1+1}$.

837. $\dfrac{e^p}{e^p-e} - \dfrac{2e^p}{e^p-\sqrt{e}}$.

838. $\dfrac{1}{2}\dfrac{e^p}{e^p-1} + \dfrac{1}{2}\dfrac{e^p(e^p-\cos 2)}{e^{2p}-2e^p\cos 2+1}$. 839. $\dfrac{e^{(1-k)p}}{e^p-1}$.

840. $\left(\dfrac{e^p}{e^p-e^3} - 1 - e^{\alpha-p} - e^{2\alpha-2p}\right) e^{3p}$.

841. $\dfrac{e^p(2\cosh 2 - e^p)\sinh 2}{e^{2p}-2e^p\cosh 2+1}$.

842. $\dfrac{e^p(4e^{2p}-3e^p+1)}{(e^p-1)^3}$. 843. $\dfrac{e^{p-1}\sin 2}{e^{2p}-2e^{p-1}\cos 2+e^{-2}}$.

844. $\dfrac{e^{p+2}(e^p+e^2)}{(e^p-e^2)^3}$. 845. $\dfrac{e^p(e^p-e^3\cosh 1)}{e^{2p}-2e^{p+3}\cosh 1+e^6}$.

846. $\dfrac{e^{p+1}(e^p+e)}{(e^p-e)^3}$. 847. $\dfrac{e^p(e^p+1)}{(e^p-1)^3}$.

848. $\dfrac{(e^{2p}-1)e^p}{(e^{2p}+1)^2}$. 849. $\ln a + \ln \dfrac{e^p-1}{e^p-a}$.

850. $\alpha + \arctan \dfrac{\sin\alpha}{e^p-\cos\alpha}$.

851. $\ln \dfrac{\sqrt{e^{2p}-2e^p\cos\alpha+1}}{e^p-1}$.

852. $\dfrac{1}{e^p-1} - \ln \sqrt{\dfrac{e^p-e^{-1}}{e^p-e}}$.

853. $\dfrac{e^p[(1+e^{2p})\cos\alpha - 2e^p]}{(e^{2p}-2e^p\cos\alpha+1)^2}$.

854. $\dfrac{e^p\,(e^{4p}+2e^{3p}\cosh\alpha-6e^{2p}+2e^p\cosh\alpha+1)\sinh\alpha}{(e^{2p}-2e^p\cosh\alpha+1)^3}$.

855. $\dfrac{e^p\,(2e^{3p}-5e^{2p}\cosh\alpha+4e^p\cosh^2\alpha-\cosh\alpha)}{(e^{2p}-2e^p\cosh\alpha+1)^2}$.

856. $\varepsilon-\varepsilon_0+\ln\dfrac{e^p-e^{\varepsilon_0}}{e^p-e^{\varepsilon}}$.

857. $\dfrac{1}{2}\ln\dfrac{e^{2p}-2e^p\cosh 1+1}{e^{2p}-2e^p\cosh\varepsilon+1}$.

858. $\varepsilon+\operatorname{arc\,tan}\dfrac{\sin\varepsilon}{e^p-\cos\varepsilon}$.

859. $\varepsilon-1+\dfrac{1}{2}\left(\operatorname{arc\,tan}\dfrac{\sin 2\varepsilon}{e^p-\cos 2\varepsilon}-\operatorname{arc\,tan}\dfrac{\sin 2}{e^p-\cos 2}\right)$.

860. $\dfrac{1-e^n}{1-e}$. 861. $\dfrac{n(n+1)}{2}$. 862. $\dfrac{n}{1-e}-\dfrac{1-e^n}{(1-e)^2}$.

863. $\dfrac{1-e^{2(n-1)}}{1-e^2}$.

864. $\Delta f(n)=3$, $\Delta^k f(n)=0$ $(k=2,\ 3,\ 4,\ \ldots)$.

865. $\Delta^k f(n)=(e^4-e^2)^k$ $(k=1,\ 2,\ 3,\ \ldots)$.

866. $\Delta f(n)=2n$, $\Delta^2 f(n)=2$, $\Delta^k f(n)=0$ $(k=3,\ 4,\ \ldots)$.

867. $\dfrac{e^p(e^{2p}+4e^p+1)}{(e^p-1)^4}$. 868. $\dfrac{e^p}{(e^p-1)^3}$.

869. $\dfrac{2e^{(1-k)p}}{(e^p-1)^3}$. 870. $\dfrac{n(n-1)(2n-1)}{6}$.

871. $\dfrac{(n-1)\sin\dfrac{2n-1}{2}\alpha}{2\sin\dfrac{\alpha}{2}}-\dfrac{\sin^2\dfrac{n-1}{2}\alpha}{2\sin^2\dfrac{\alpha}{2}}$.

872. $\dfrac{n(n-1)(n-2)}{3}$.

873. $\dfrac{(1-e\cos\alpha)(1-e^n\cos n\alpha)+e^{n+1}\sin\alpha\sin\alpha n}{e^2-2e\cos\alpha+1}$.

874. $n3^{n-1}$. 875. $\dfrac{5^n-3^n}{3}$. 876. $\sin\dfrac{n\pi}{2}$.

877. $\dfrac{1+(-1)^n}{4}-\dfrac{1}{2}\sin\dfrac{n+1}{2}\pi$.

878. $a^{n-1}2^{n/2}\sin\dfrac{3n\pi}{4}$. 879. $\dfrac{n(n-1)}{2}e^{n-2}$.

880. Second-order equation.
881. Zeroth-order equation.
882. First-order equation.
883. Third-order equation.

884. 2^n. 885. $(-1)^n(1-n)$. 886. $2^{n/2}\sin\frac{n\pi}{4}$.

887. $1-2^{n/2}\cos\frac{n\pi}{4}$ 888. $\sin\frac{n\pi}{2}\cos\frac{(n-1)\pi}{4}$.

889. $(-1)^n\frac{n^2-n}{2}$ 890. $\frac{3n-1+(-2)^n}{9}$

891. $\frac{4^{n-1}+15\times 2^{n-3}-7(-2)^{n-3}}{3}$ 892. $\frac{1-(-1)^n}{2}$.

893. $\frac{n(n-1)(n-2)}{2}3^{n-3}$.

894. $\frac{n(n-1)(n-2)}{6}(-1)^{n-1}$.

895. $\frac{n^5}{60}-\frac{n^4}{8}+\frac{n^3}{3}-\frac{3n^2}{8}+\frac{3n}{20}$.

896. Asymptotically stable.
897. Asymptotically stable.
898. Unstable. 899. Unstable.
900. Stable but not asymptotically.
901. Asymptotically stable. 902. Unstable.
903. Unstable focal point. 904. Vortex point.
905. Stable focal point. 906. Saddle point.
907. Unstable nodal point. 908. Unstable nodal point.
909. Stable nodal point.
910. Point (0, 0, 0) is stable.
911. Point (0, 0, 0) is unstable.
912. Asymptotically stable for $\alpha < 0$ and unstable in all other cases.
913. Asymptotically stable for $\alpha < 0$ and $\alpha > 1$; stable but not asymptotically for $\alpha = 0$ and $\alpha = 1$; unstable for $0 < \alpha < 1$.
914. Unstable for all values of α. 915. $\alpha \leqslant 0$.
916. $\alpha \leqslant -1/2$.
917. Asymptotically stable for $\alpha\beta < 1$; stable but not asymptotically for $\alpha\beta = 1$.
918. Asymptotically stable for $\beta < \alpha^2$ ($\alpha < 0$); stable but not asymptotically for (1) $\alpha = 0$ ($\beta < 0$) and (2) $\beta = \alpha^2$ ($\alpha < 0$).
919. Asymptotically stable for $\alpha^2 + \beta^2 - 2\alpha > 0$ ($\alpha < 1$); stable but not asymptotically for (1) $\alpha = 1$ ($|\beta| > 1$) and (2) $\alpha^2 + \beta^2 - 2\alpha = 0$ ($0 \leqslant \alpha < 1$).
920. Unstable for all values of α and β.
921. Asymptotically stable for $\alpha^2 + \beta^2 - \beta < 0$; stable but not asymptotically for $\alpha^2 + \beta^2 - \beta = 0$ ($\alpha \neq 0$ and $\beta \neq 0$).
922. Stable but not asymptotically for $\beta + 2\alpha + 1 = 0$; asymptotically stable for all other values of α and β.
923. Asymptotically stable.

924. Asymptotically stable.
925. Asymptotically stable. 926. Stable.
927. Asymptotically stable.
928. Asymptotically stable.
929. Asymptotically stable.
930. Unstable.
931. Unstable.
932. Asymptotically stable.
933. Asymptotically stable.
934. Asymptotically stable.
935. Unstable. 936. Stable. 937. Unstable.
938. Stable. 939. Unstable. 940. Unstable.
941. Asymptotically stable. 942. Stable.
943. Investigation for stability in the first approximation is impossible. Employing a Lyapunov function, we can establish that point (0, 0) is asymptotically stable.
944. The critical point is stable.
945. The zero solution of the first-order approximation system is unstable, but for the original system it is asymptotically stable.
946. If $a > 0$ and $b > 0$, then the stability criterion has the form $\cos T > 0$ where $T = (-1)^k x_0 + k\pi$ $(k = 0, 1, 2, \ldots)$,

$$x_0 = \arcsin (L/b)$$

(see *Barbashin*).
954. Stable. 955. Unstable. 956. Stable. Л
957. Unstable. 958. Stable for $\alpha > 1/2$.
959. Unstable for all values of α.
960. Stable for $\alpha > 13/6$.

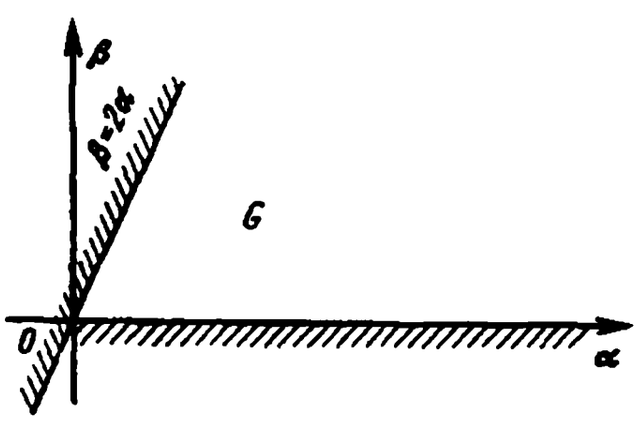


Figure to answer to Problem 961

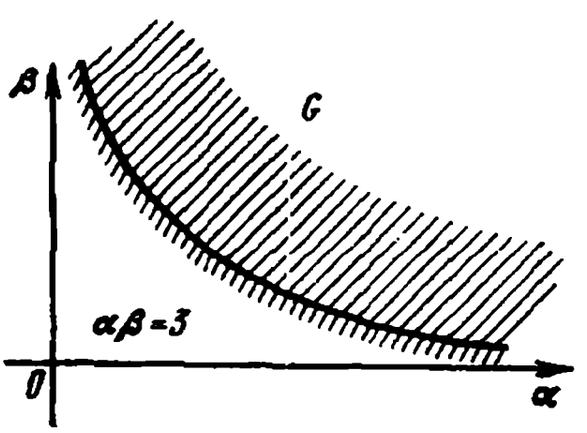


Figure to answer to Problem 962

961. Stable for all values of α and β that belong to the region G (see the figure).
962. Stable for all values of α and β that belong to the region G: $\alpha\beta > 3$, $\alpha > 0$, $\beta > 0$ (see the figure).

963. Unstable for all values of α and β.

964. $p > 0$, $q > 2$.

965. All roots lie in the left half-plane; the solution is stable (see the figure).

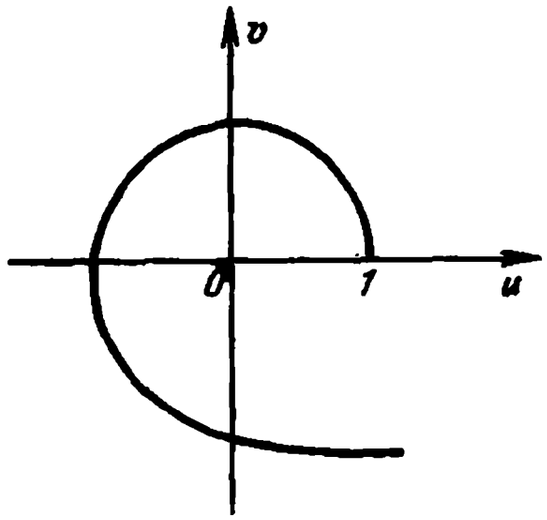


Figure to answer to Problem 965

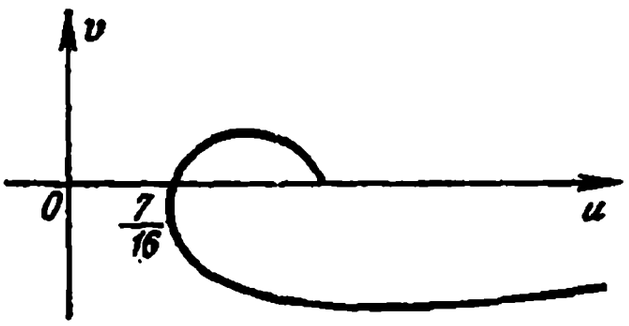


Figure to answer to Problem 966

966. Two roots lie in the left half-plane and two in the right half-plane; the solution is unstable (see the figure).

967. Stable. 968. Stable.

969. Two roots lie in the right half-plane; the solution is unstable see the figure).

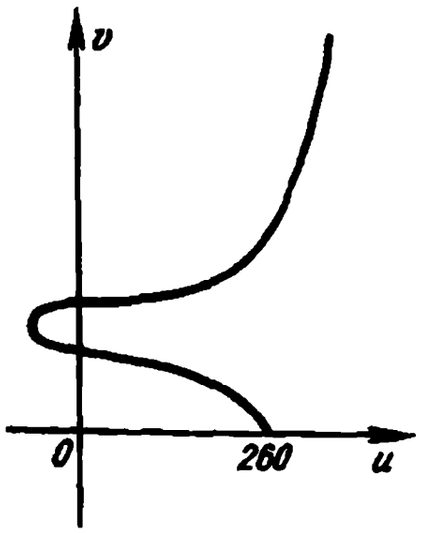


Figure to answer to Problem 969

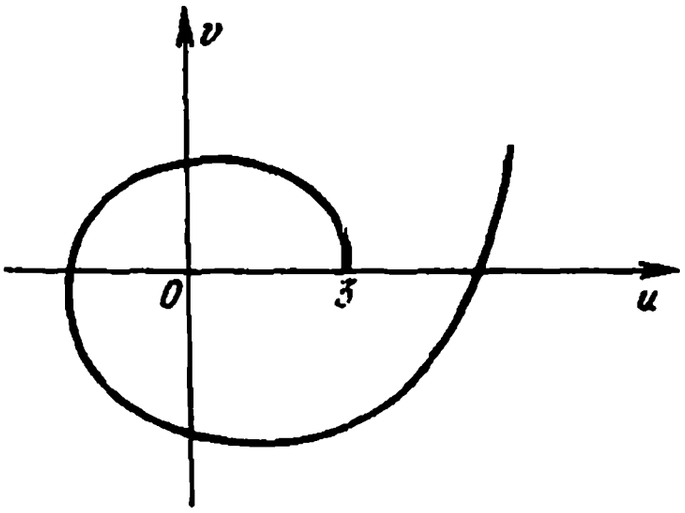


Figure to answer to Problem 970

970. Stable (see the figure). 971. Stable.

972. Stable. 973. Stable. 974. Stable.

975. Stable. 976. Stable. 977. Stable.

978. Stable. 979. Stable (see the figure).
980. Stable.
981. Purely imaginary roots; the solution is unstable (see the figure).

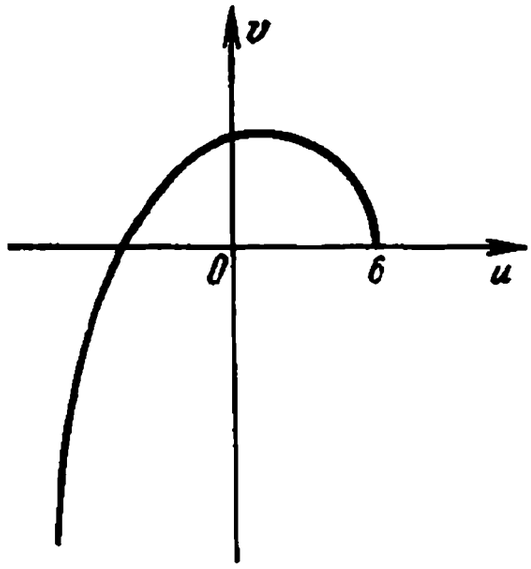


Figure to answer to Problem 979

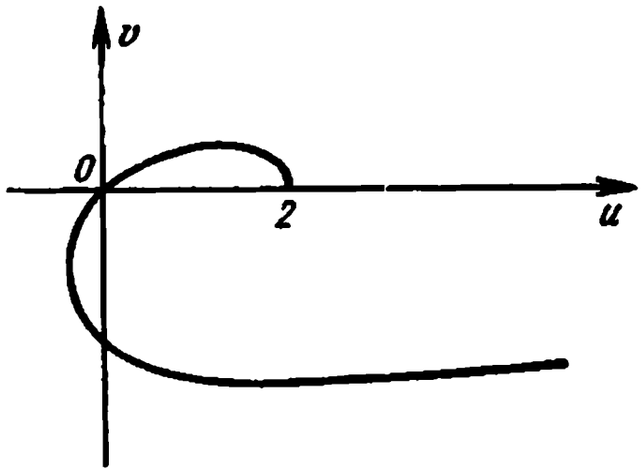


Figure to answer to Problem 981

982. Two roots in the right half-plane; the solution is unstable.
983. Two roots in the right half-plane; the solution is unstable (see the figure).

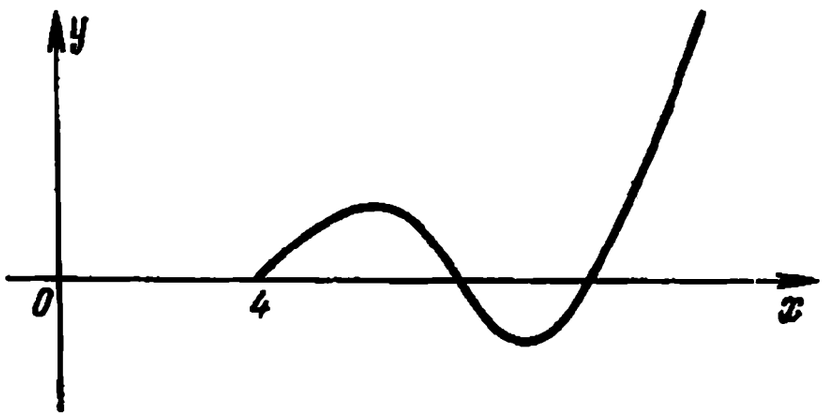


Figure to answer to Problem 983

984. Two roots in the right half-plane; the solution is unstable.

985.

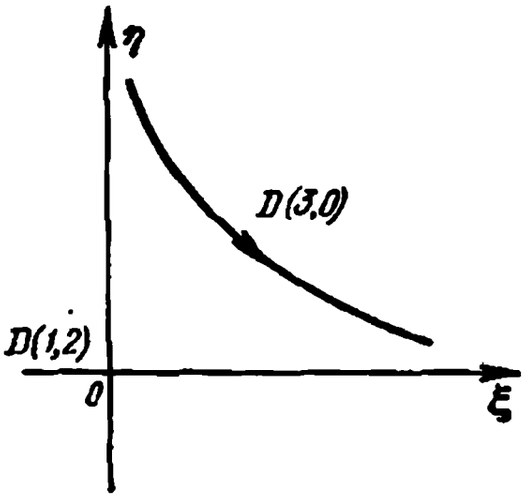


Figure to answer to Problem 985

986.

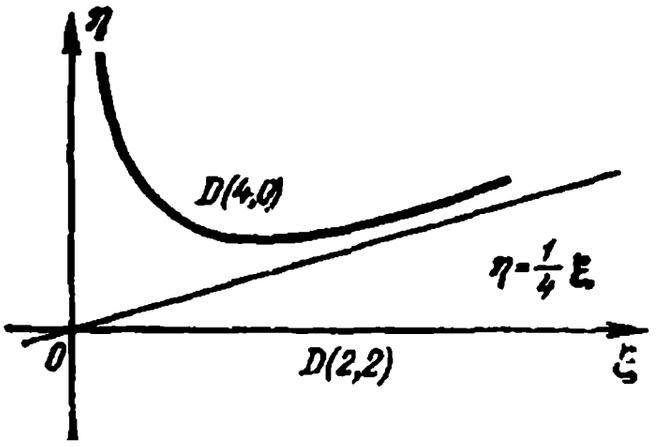


Figure to answer to Problem 986

987.

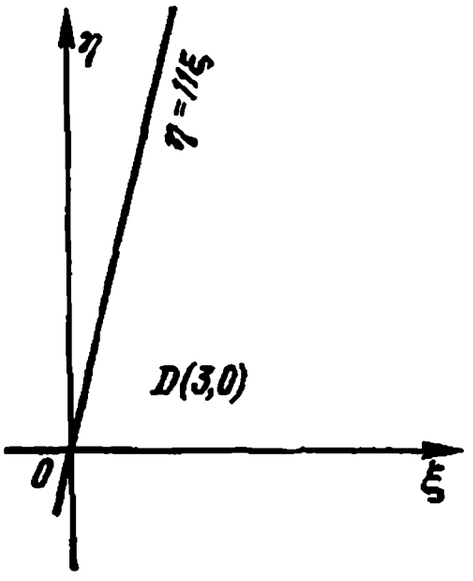


Figure to answer to Problem 987

988.

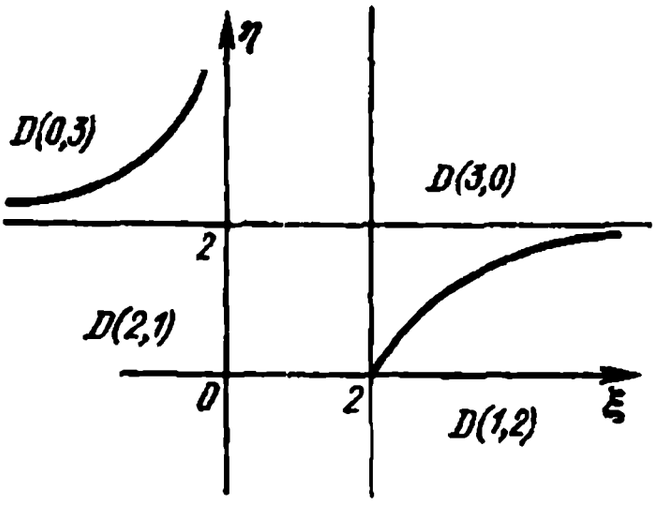


Figure to answer to Problem 988

989.

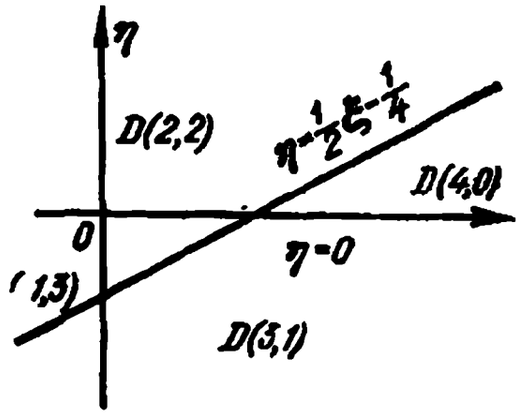


Figure to answer to Problem 989

990.

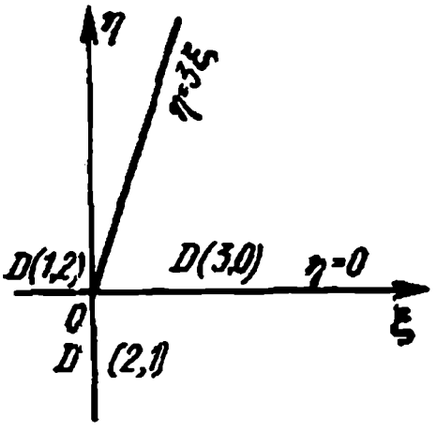


Figure to answer to Problem 990

991.

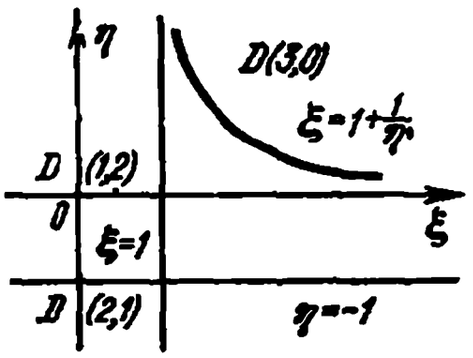


Figure to answer to Problem 991

992.

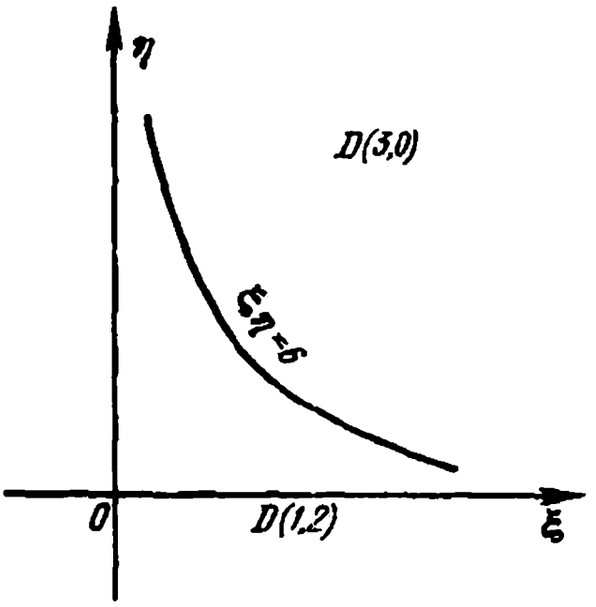


Figure to answer to Problem 992

993.

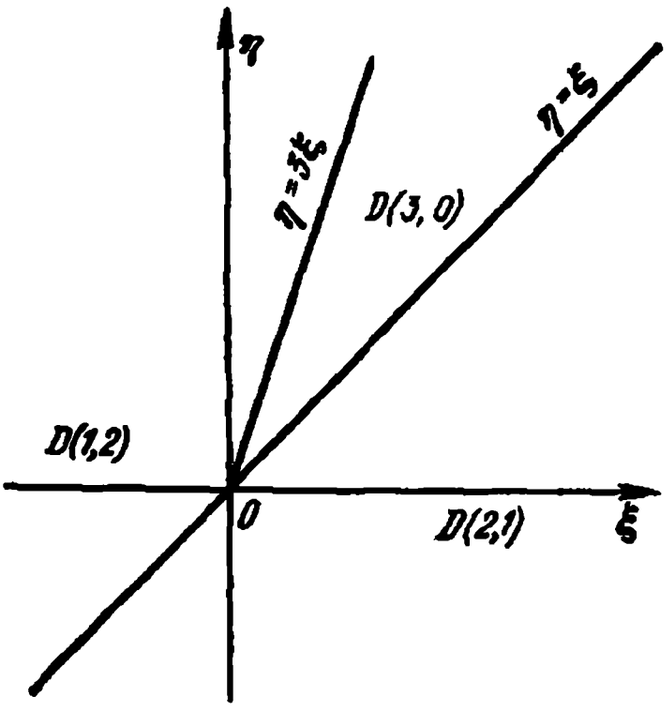


Figure to answer to Problem 993

994.

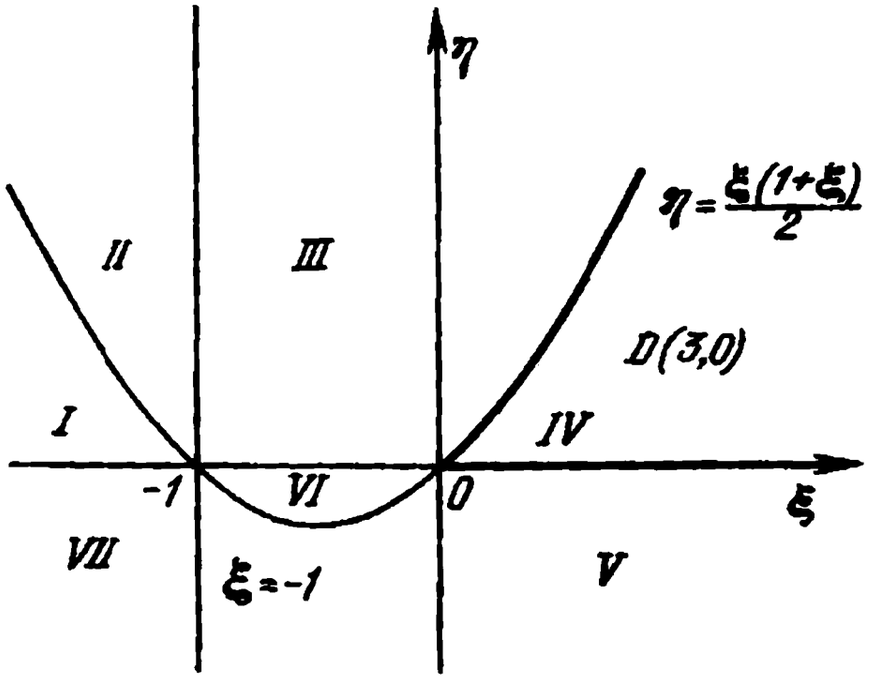


Figure to answer
to Problem 994

995.

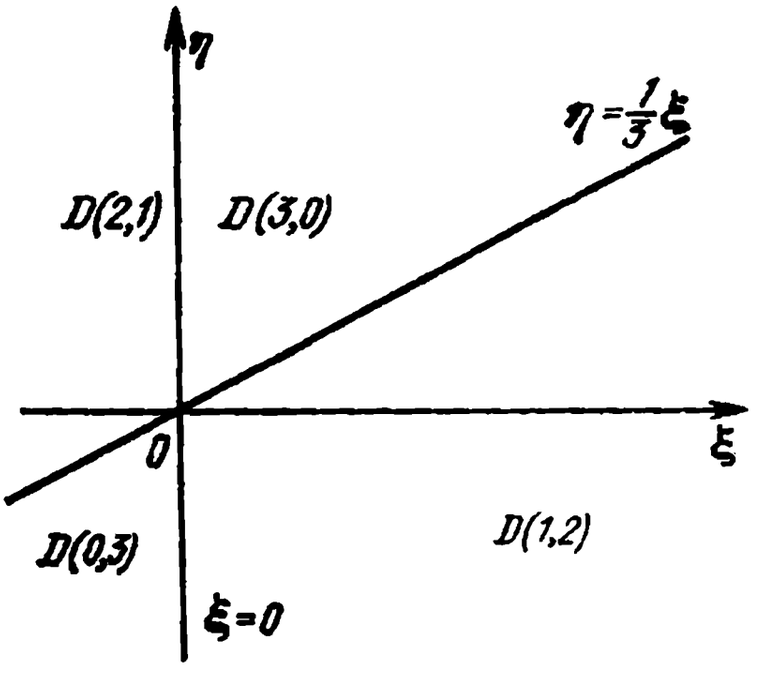


Figure to answer
to Problem 995

996.

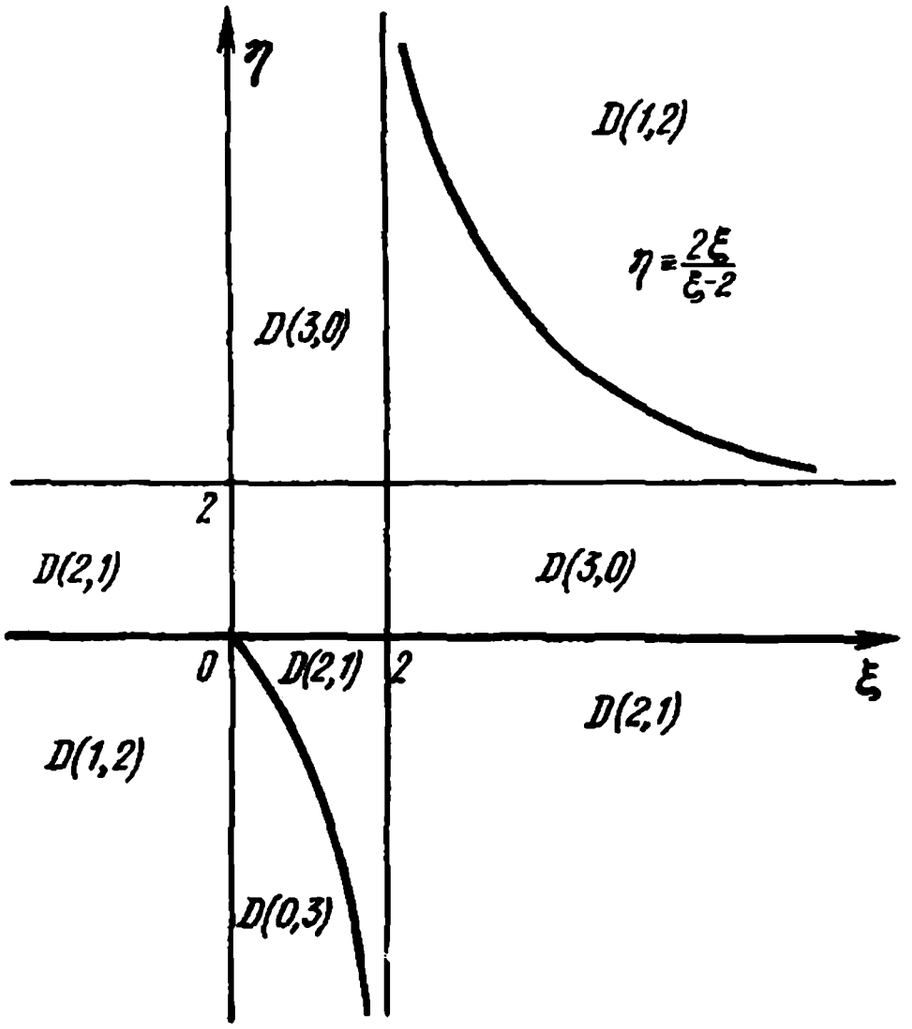


Figure to answer to Problem 996

997.

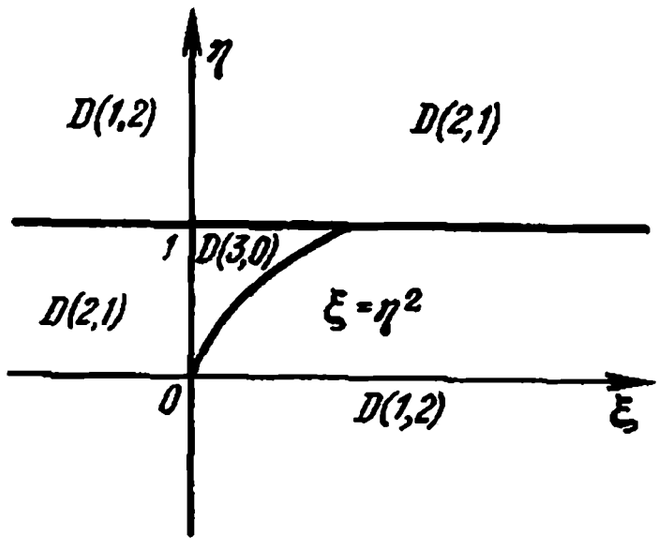


Figure to answer to Problem 997

998. $f(n)=C_1 2^n+C_2\left(-\frac{4}{5}\right)^n$.

999. $f(n)=(-1)^n(4n^2-7n+1)$.

1000. $f(n)=\left(\frac{\sqrt{5}}{2}\right)^n\left[C_1\cos\left(n\arctan\frac{1}{2}\right)+C_2\sin\left(n\arctan\frac{1}{2}\right)\right]$.

1001. $f(n)=2^n\left(C_1+C_2\cos\frac{2n\pi}{3}+C_3\sin\frac{2n\pi}{3}\right)$.

1002. $f(n)=(-1)^n(C_1+C_2n)+2^{n/2}\left(C_3\cos\frac{n\pi}{4}+C_4\sin\frac{n\pi}{4}\right)$.

1003. $f(n)=C_1(1-\sqrt{2})^n+C_2(1+\sqrt{2})^n-\frac{n}{2}$.

1004. $f(n)=2\times 3^n+(-1)^n(8n-2)$.

1005. $f(n)=\frac{1}{2}\tan 2\times\cos\frac{n\pi}{2}+\sin\frac{n\pi}{2}+\frac{\sin 2(n-1)}{2\cos 2}$.

1006. $f(n)=C_1+C_2n+C_3n^2+\frac{e^n}{(e-1)^3}$.

1007. $f(n)=2^n\left(\frac{1}{16}+C_1\cos\frac{n\pi}{3}+C_2\sin\frac{n\pi}{3}\right)+C_3(-2)^n$.

1008. Asymptotically stable.
1009. Stable but not asymptotically.
1010. Asymptotically stable. 1011. Unstable.
1012. $a_0+a_1+a_2+a_3>0$, $a_0-a_1+a_2-a_3>0$, $3(a_0-a_3)+a_1-a_2>0$, $3(a_0+a_3)-a_1-a_2>0$, $a_0^2-a_3^2-a_0a_2+a_1a_3>0$.
1013. $1-q>0$, $1+p+q>0$, $1-p+q>0$.
1014. $-1<p<1$. 1015. $|a|>b$.
1016. Asymptotically stable. 1017. Unstable.
1018. Asymptotically stable. 1019. Unstable.
1020. Unstable. 1021. Unstable. 1022. Unstable.

Appendix

Object and Result Functions

Object function $f(t)$	Result function $F(p) = \int_0^\infty f(t)\, e^{-pt}\, dt$
1	$\frac{1}{p}$
$t^n\ (n = 1, 2, \ldots)$	$\frac{n!}{p^{n+1}}$
$t^\alpha\ (\alpha > -1)$	$\frac{\Gamma(\alpha + 1)}{p^{\alpha+1}}$
$e^{\lambda t}\ (\lambda = a + bi)$	$\frac{1}{p - \lambda}$
$t^n e^{\lambda t}$	$\frac{n!}{(p-\lambda)^{n+1}}$
$t^\alpha e^{\lambda t}\ (\alpha > -1)$	$\frac{\Gamma(\alpha + 1)}{(p-\lambda)^{\alpha+1}}$
$\sin \omega t\ (\omega > 0)$	$\frac{\omega}{p^2 + \omega^2}$
$\cos \omega t$	$\frac{p}{p^2 + \omega^2}$
$\sinh \omega t$	$\frac{\omega}{p^2 - \omega^2}$
$\cosh \omega t$	$\frac{p}{p^2 - \omega^2}$

(Continued)

Object function $f(t)$	Result function $F(p) = \int_0^\infty f(t)\, e^{-pt}\, dt$
$e^{\lambda t} \sin \omega t$	$\dfrac{\omega}{(p-\lambda)^2+\omega^2}$
$e^{\lambda t} \cos \omega t$	$\dfrac{p-\lambda}{(p-\lambda)^2+\omega^2}$
$e^{\lambda t} \sinh \omega t$	$\dfrac{\omega}{(p-\lambda)^2-\omega^2}$
$e^{\lambda t} \cosh \omega t$	$\dfrac{p-\lambda}{(p-\lambda)^2-\omega^2}$
$t \sin \omega t$	$\dfrac{2p\omega}{(p^2+\omega^2)^2}$
$t \cos \omega t$	$\dfrac{p^2-\omega^2}{(p^2+\omega^2)^2}$
$t \sinh \omega t$	$\dfrac{2p\omega}{(p^2-\omega^2)^2}$
$t \cosh \omega t$	$\dfrac{p^2+\omega^2}{(p^2-\omega^2)^2}$
$\sin (t-\tau)\ (\tau > 0)$	$\dfrac{e^{-\tau p}}{p^2+1}$
$\cos (t-\tau)$	$\dfrac{pe^{-\tau p}}{p^2+1}$
$t^n \sin \omega t$	$\dfrac{\operatorname{Im} (p+i\omega)^{n+1}}{(p^2+\omega^2)^{n+1}}\, n!$
$t^n \cos \omega t$	$\dfrac{\operatorname{Re} (p+i\omega)^{n+1}}{(p^2+\omega^2)^{n+1}}\, n!$
$J_n(t)\ (n=1, 2, \ldots)$	$\dfrac{(\sqrt{p^2+1}-p)^n}{\sqrt{p^2+1}}$
$\operatorname{Erf}\left(\dfrac{\alpha}{2\sqrt{t}}\right) = \dfrac{2}{\sqrt{\pi}} \int_{\alpha/2\sqrt{t}}^\infty e^{-x^2}\, dx$	$\dfrac{1}{p} e^{-\alpha \sqrt{p}}$

(Continued)

Object function $f(t)$	Result function $F(p) = \int_0^\infty f(t)\, e^{-pt}\, dt$
$\operatorname{Si} t = \int_0^t \frac{\sin x}{x}\, dx$	$\frac{\operatorname{arc\,cot} p}{p}$
$\operatorname{Ci} t = -\int_t^\infty \frac{\cos x}{x}\, dx$	$\frac{1}{p} \ln \frac{1}{\sqrt{p^2+1}}$
$\frac{e^{bt} - e^{at}}{t}$	$\ln \frac{p-a}{p-b}$
$\frac{I_n(\alpha t)}{t}$ $(\alpha > 0)$	$\frac{1}{n\alpha^n} (\sqrt{p^2+\alpha^2} - p)^n$
$\ln t$	$\frac{1}{p}\left(\ln \frac{1}{p} - C\right)$, $C \approx 0.577216$

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